

# MULTIPLE SCATTERING

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### 1. INTRODUCTION

Considerable simplifications in the theory describing the reactions induced by a projectile incident on a nucleus become possible when the energy of the projectile is sufficiently high. In this limit, the projectile in passing through the nucleus can be considered to undergo successive collisions with the target nucleons—hence the term *multiple scattering*. In the lowest approximation, each of these collisions is treated as a two-body (projectile–target nucleon) collision. One can thereby relate the transition amplitude for the processes induced by the collision with the nucleus to the transition amplitude for those induced by the collision of the projectile with the individual nucleons making up the nucleus. The complex many-body problem is thus reduced to a simpler two-body one, although it must be borne in mind that the projectile–nucleon collision occurs in an “environment” produced by the other nucleons in the nucleus, sometimes referred to as *spectators*.

For this approximation to be accurate, it is necessary for the nucleus to be sufficiently dilute so that the projectile encounters only one target nucleon at a time. The range of the force between the projectile and nucleon should therefore be small compared to the distance  $2r_0$  between the nucleons. A second condition requires that the wavelength of the projectile be small compared to the distance between the nucleons of the nucleus:

$$kr_0 \gg 1 \tag{1.1}$$

where  $k = 1/\lambda$ . If the wavelength is so long that this condition is not satisfied,

a collision with one nucleon will necessarily involve its neighbors and thus it cannot be regarded as a two-body collision. It will be recognized that an essential part of these conditions is the requirement that the projectile is "on the energy shell" between collisions; that is, between collisions the energy is kinetic,  $\hbar^2 k^2/2m$ . One can readily take into account a *constant* potential energy,  $-V$ , in the region between collisions where the energy is then  $(\hbar^2 k^2/2m) + V$ .

Under these circumstances the projectile wave proceeds through the nucleus, producing at each target nucleon a scattered wave. These scattered waves for elastic scattering will be coherent and in the forward direction will interfere constructively, leading to a scattering amplitude proportional to the number of nucleons,  $A$ , in the target. The cross section in the forward direction will be proportional to  $A^2$ . For larger angles of scattering the cross section will decrease as the angle of scattering or equivalently the momentum transfer,  $q$ , increases. The relevant parameter can be determined by the following considerations.

When an incident projectile collides elastically with a target nucleon, it will impart a momentum transfer  $q$  and therefore an energy  $\hbar^2 q^2/2M$  to the target nucleon. However, at small angles, that energy is not sufficient to lift the nucleon out of the Fermi sea. There must be further collisions of the struck target nucleon with the other nucleons, so that finally the nucleus recoils as a whole. For that to occur, the uncertainty in position of the struck nucleon,  $\hbar/q$  must be larger than the size of the nucleus. Hence, roughly

$$\frac{\hbar}{q} > R \quad (1.2)$$

where  $R$  is the nuclear radius. Or since for elastic scattering

$$q = 2p \sin \frac{1}{2}\theta$$

inequality (1.2) becomes

$$2kR \sin \frac{\theta}{2} < 1 \quad (1.3)$$

For angles greater than those satisfying (1.3),  $\hbar/q$  will be less than  $R$  and only part of the nucleus will be involved. Thus the nucleus will not be able to recoil as a whole and the elastic scattering amplitude will be reduced, falling from its  $\theta = 0$  value.

When these conditions are satisfied, one can, for example, obtain the elastic scattering amplitude for the collision of the projectile by the target nucleus in terms of the nuclear density, and of the two-body (projectile–nucleon) scattering amplitude. This result is of great power since it permits the determination of the nuclear density from experiments using a variety of projectiles that probe different spin-, isospin-, and momentum-dependent components of the nuclear density.

The importance of the approximations involved must, of course, be evaluated. A detailed discussion is given in Section 4. For the present we note that the conditions described above are never exactly satisfied since there is a finite probability that the separation of two target nucleons will be so small that the potentials between the projectile and the target nucleons will overlap and/or that the projectile wavelength will not be sufficiently small. In that event the projectile will interact with at least two of the target nucleons simultaneously. The transition amplitude for projectile–nucleus reactions will then depend not only on the density of target nucleons but also on their spatial correlation. One such correlation is induced by the Pauli exclusion principle and has the scale given by  $1/k_F$ , so that nucleons separated by less than this distance can no longer be considered as independent [see Chapter III in deShalit and Feshbach (74)]. Other correlations will be a consequence of the nuclear forces acting between the target nucleons. The scales in this case are the various ranges characterizing the nucleon–nucleon potential. The strength of, for example, the pair correlation is given by the number of target nucleons, multiplied by the probability of a nucleon finding another within a distance  $r_c$  ( $\equiv$  scale length of the correlation). The latter factor is given by  $(r_c/R)^3$ , so that the correlation effect is of the order of

$$A\left(\frac{r_c}{R}\right)^3 = \left(\frac{r_c}{r_0}\right)^3 \quad (1.4)$$

using  $R = r_0 A^{1/3}$ . It is thus difficult to observe correlation lengths much smaller than  $r_0$ . (For a more quantitative result, see Appendix A.)

Another approximation we shall often use in this chapter asserts that the projectile passes through the nucleus in so short a time that the target nucleons are essentially stationary. This approximation, referred to as the *frozen nucleus approximation*, is valid for sufficiently high projectile energy. The characteristic time,  $\tau_n$ , for target nucleon motion can be obtained from the zero-point motion of the nucleon of amplitude  $r_0$ . The corresponding momentum is  $\hbar/r_0$  with energy  $\Delta E = \hbar^2/2Mr_0^2$ . The time  $\tau_n$  is then

$$\tau_n \sim \frac{\hbar}{\Delta E} = \frac{2Mr_0^2}{\hbar} \quad (1.5)$$

The time required for the projectile to pass a target nucleon is

$$\tau_p = 2r_0 \frac{E_p}{p_p c^2} \quad (1.6)$$

where  $E_p$  and  $p_p$  are the energy, including the rest mass and momentum of the projectile, respectively. The ratio  $\tau_n/\tau_p$  needs to be much greater than 1 if the

frozen approximation is to be valid:

$$\frac{\tau_n}{\tau_p} = k_p r_0 \frac{Mc^2}{E_p} \quad (1.7)$$

In the limit  $E_p \gg M_p c^2$ , this ratio becomes  $Mc^2 r_0 / \hbar c$ , which is much larger than unity. In the nonrelativistic limit, one obtains for the ratio,  $k_p r_0 M / M_p$ . Thus if the projectile is a nucleon, inequality (1.7) becomes identical with (1.1).

Under the frozen nucleus approximation the transition amplitude  $\hat{\mathcal{F}}$  is a function of the position, spin, and so on, of each of the target nucleons that prevail at the time the projectile passes through the target:

$$\hat{\mathcal{F}} = \hat{\mathcal{F}}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A) \quad (1.8)$$

The transition amplitude to be compared with experiment is obtained by taking appropriate matrix elements of  $\hat{\mathcal{F}}$  with respect to the target nucleus states. For elastic scattering that amplitude is

$$\mathcal{F}_{ii} = \langle \Psi_i | \hat{\mathcal{F}} | \Psi_i \rangle \quad (1.9)$$

where  $\Psi_i$  is the target nucleus wave function. For inelastic scattering, it is

$$\mathcal{F}_{fi} = \langle \Psi_f | \hat{\mathcal{F}} | \Psi_i \rangle \quad (1.10)$$

where  $\Psi_f$  is the final target nucleus state. This approximation is called *adiabatic*.

## 2. QUALITATIVE RESULTS†

Much of the physics of multiple scattering can be understood at a qualitative level by studying a simple case. In this example the target nucleus is taken to be a system of nucleons which is so dilute that the probability the projectile will undergo two collisions with a target nucleon is small. The frozen nucleus approximation will be used so that the target nucleons will be considered as being fixed during the course of the collision.

Under these circumstances the transition amplitude  $\hat{\mathcal{F}}$  will be given by a sum of amplitudes emanating from each of the target nucleons:

$$\hat{\mathcal{F}} = \sum_n t(\mathbf{k}_f, \mathbf{k}_i; \mathbf{r}_n) \quad (2.1)$$

where  $t$  is the projectile–nucleon amplitude giving the scattering of a projectile with incident momentum  $\hbar \mathbf{k}_i$  and final momentum  $\hbar \mathbf{k}_f$  by a target nucleon at  $\mathbf{r}_n$ .

†Lax (51).

We now must make use of an important theorem relating the amplitude  $t(\mathbf{k}_f, \mathbf{k}_i, \mathbf{r}_n)$  with the amplitude of a scatterer located at the origin  $t(\mathbf{k}_f, \mathbf{k}_i; 0)$ . Toward this end, compare the integral Schrödinger equations appropriate to these two situations:

$$\psi(\mathbf{r}; \mathbf{r}_n) = e^{i\mathbf{k}_i \cdot \mathbf{r}} - \frac{1}{4\pi} \int \frac{e^{i\mathbf{k}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}' - \mathbf{r}_n) \psi(\mathbf{r}'; \mathbf{r}_n) d\mathbf{r}' \quad (2.2a)$$

$$\psi(\mathbf{r}; 0) = e^{i\mathbf{r}_i \cdot \mathbf{r}} - \frac{1}{4\pi} \int \frac{e^{i\mathbf{k}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}') \psi(\mathbf{r}'; 0) d\mathbf{r}' \quad (2.2b)$$

where  $V$  is the scattering potential. In the first of these equations introduce a shift of the origin:

$$\mathbf{r}' - \mathbf{r}_n = \mathbf{s}' \quad \mathbf{r} - \mathbf{r}_n = \mathbf{s}$$

One obtains

$$\psi(\mathbf{s} + \mathbf{r}_n; \mathbf{r}_n) = e^{i\mathbf{k}_i \cdot \mathbf{s}} e^{i\mathbf{k}_i \cdot \mathbf{r}_n} - \frac{1}{4\pi} \int \frac{e^{i\mathbf{k}|\mathbf{s}-\mathbf{s}'|}}{|\mathbf{s}-\mathbf{s}'|} V(\mathbf{r}') \psi(\mathbf{s}' + \mathbf{r}_n; \mathbf{r}_n) d\mathbf{s}'$$

Comparing this equation with (2.2b) yields immediately the important result

$$\psi(\mathbf{s} + \mathbf{r}_n; \mathbf{r}_n) = e^{i\mathbf{k}_i \cdot \mathbf{r}_n} \psi(\mathbf{s}; 0) \quad (2.3)$$

Thus the shift in the origin results only in a change in phase, a consequence of translational invariance of the Schrödinger equation (2.2a). From (2.2a) we obtain

$$\begin{aligned} t(\mathbf{k}_f, \mathbf{k}_i; \mathbf{r}_n) &= \int e^{-i\mathbf{k}_f \cdot \mathbf{r}'} V(\mathbf{r}' - \mathbf{r}_n) \psi(\mathbf{r}'; \mathbf{r}_n) d\mathbf{r}' \\ &= e^{-i\mathbf{k}_f \cdot \mathbf{r}_n} \int e^{-i\mathbf{k}_f \cdot \mathbf{s}'} V(\mathbf{s}') \psi(\mathbf{s}' + \mathbf{r}_n; \mathbf{r}_n) d\mathbf{s}' \end{aligned}$$

Using (2.3) it follows that

$$t(\mathbf{k}_f, \mathbf{k}_i; \mathbf{r}_n) = e^{i(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{r}_n} t(\mathbf{k}_f, \mathbf{k}_i; 0) \quad (2.4)$$

One corollary of this equation will be important for later discussion. We leave it as a problem.

**Problem.** Let  $t(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n)$  be defined by

$$t(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n) \equiv \iint e^{i\mathbf{k}_f \cdot \mathbf{r}} t(\mathbf{k}_f, \mathbf{k}_i; \mathbf{r}_n) e^{-i\mathbf{k}_i \cdot \mathbf{r}'} \frac{d\mathbf{k}_f}{(2\pi)^3} \frac{d\mathbf{k}_i}{(2\pi)^3} \quad (2.5a)$$

Show that

$$t(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n) = t(\mathbf{r} - \mathbf{r}_n, \mathbf{r}' - \mathbf{r}_n; 0) \quad (2.5b)$$

We can now return to (2.1). Using (2.4), it becomes

$$\begin{aligned} \hat{\mathcal{F}} &= t(\mathbf{k}_f, \mathbf{k}_i) \sum_n e^{i\mathbf{q} \cdot \mathbf{r}_n} \\ \mathbf{q} &\equiv \mathbf{k}_i - \mathbf{k}_f \\ t(\mathbf{k}_f, \mathbf{k}_i) &\equiv t(\mathbf{k}_f, \mathbf{k}_i; 0) \end{aligned} \quad (2.6)$$

Finally, the elastic scattering transition amplitude is

$$\mathcal{F}_{el}(\mathbf{k}_f, \mathbf{k}_i) = t(\mathbf{k}_f, \mathbf{k}_i) \langle \Psi_i | \sum_n e^{i\mathbf{q} \cdot \mathbf{r}_n} \Psi_i \rangle$$

Using the antisymmetry of the wave function  $\Psi_i$ , this becomes

$$\mathcal{F}_{el}(\mathbf{k}_f, \mathbf{k}_i) = A t(\mathbf{k}_f, \mathbf{k}_i) \tilde{\rho}(\mathbf{q}) \quad (2.7)$$

where

$$\tilde{\rho}(\mathbf{q}) = \int e^{i\mathbf{q} \cdot \mathbf{r}_1} \rho(\mathbf{r}_1) d\mathbf{r}_1$$

and

$$\rho(\mathbf{r}_1) = \int |\Psi_i(\mathbf{r}_1, \mathbf{r}_2, \dots)|^2 d\mathbf{r}_2 \dots \quad (2.8)$$

$$\int \rho(\mathbf{r}_1) d\mathbf{r}_1 = 1 \quad (2.9)$$

From (2.9) it follows that the Fourier transform of the density,  $\tilde{\rho}(\mathbf{q})$ , is unity at  $\mathbf{q} = 0$ , that is, for scattering in the forward direction. Generally, as described in the preceding section,  $\tilde{\rho}$  will drop rapidly with increasing scattering angle. (Note that  $|q| = 2k \sin \frac{1}{2}\theta$ , where  $\theta$  is the scattering angle.) For example, if

$$\rho(\mathbf{r}) = \left(\frac{3}{2\pi}\right)^{3/2} \frac{1}{R^3} e^{-(3/2)(r^2/R^2)} \quad (2.10)$$

which satisfies (2.9) and

$$R^2 = \int \rho(\mathbf{r}) r^2 d\mathbf{r}$$

so that  $R$  is the root-mean-square radius, then

$$\tilde{\rho}(\mathbf{q}) = e^{-q^2 R^2/6} = e^{-(2/3)k^2 R^2 \sin^2(1/2)\theta} \quad (2.11)$$

We note the rapid decrease in  $\tilde{\rho}(\mathbf{q})$  and therefore of the scattering transition amplitude, (2.7), as the scattering angle, the energy, or the size of the system increases. The quantity  $\tilde{\rho}(\mathbf{q})$  is referred to as the *form factor*.

According to (1.10), the inelastic transition amplitude is given by

$$\mathcal{F}_{fi}(\mathbf{k}_f, \mathbf{k}_i) = t(\mathbf{k}_f, \mathbf{k}_i) \langle \Psi_f | \sum_n e^{i\mathbf{q} \cdot \mathbf{r}_n} \Psi_i \rangle \quad (2.12a)$$

or

$$\mathcal{F}_{fi} = A t(\mathbf{k}_f, \mathbf{k}_i) \tilde{\rho}_{fi}(\mathbf{q}) \quad (2.12b)$$

where  $\tilde{\rho}_{fi}(\mathbf{q})$  is the Fourier transform of the transition density:

$$\tilde{\rho}_{fi}(\mathbf{q}) = \int e^{i\mathbf{q} \cdot \mathbf{r}_1} \rho_{fi}(\mathbf{r}_1) d\mathbf{r}_1 \quad (2.13)$$

and

$$\rho_{fi}(\mathbf{r}_1) = \int \Psi_f^*(\mathbf{r}_1, \mathbf{r}_2, \dots) \Psi_i(\mathbf{r}_1, \mathbf{r}_2, \dots) d\mathbf{r}_2 \dots \quad (2.14)$$

Note that  $\mathcal{F}_{fi}(\mathbf{k}_f, \mathbf{k}_i)$  is zero whenever  $q$  is zero because of the orthogonality of  $\Psi_f$  and  $\Psi_i$ . As a consequence,  $\mathcal{F}_{fi}$  at small  $q$  will be proportional to some power of  $q$ .

Further insight is obtained from examination of the total angular distribution for inelastic scattering summing over all possible final states:

$$\sum_f \frac{d\sigma_{\text{tot}}^{(\text{inel})}}{d\Omega} = \frac{2\pi}{\hbar} (\rho_f/j_i) \sum_f |\mathcal{F}_{fi}|^2 = \frac{m^2}{(2\pi)^2 \hbar^4} \frac{k_f}{k_i} \sum_f |\mathcal{F}_{fi}|^2 \quad (2.15)$$

where the projectile wave functions appearing in  $\mathcal{F}_{fi}$  are asymptotically plane waves of unit amplitude and  $m$  is the reduced mass of the projectile–nucleus system.  $\rho_f$  is the density of final states and  $j_i$  is the incident current density. Inserting (2.12a) into (2.15) and summing over the final states one obtains

$$\sum_f \frac{d\sigma_{\text{tot}}^{(\text{inel})}}{d\Omega} = \frac{m^2}{(2\pi)^2 \hbar^4} \left\langle \frac{k_f}{k_i} |t(\mathbf{k}_f, \mathbf{k}_i)|^2 \right\rangle_{\text{av}} \sum_f \langle \Psi_i | \sum_n e^{-i\mathbf{q} \cdot \mathbf{r}_n} \Psi_f \rangle \langle \Psi_f | \sum_m e^{i\mathbf{q} \cdot \mathbf{r}_m} \Psi_i \rangle \quad (2.16)$$

The factor in front of the sum is the cross section for projectile–nucleon scattering except that the projectile–nucleon reduced mass is replaced by projectile–nucleus reduced mass. We shall return to this point later.

Making use of the completeness of the final states  $\Psi_f$  the sum in (2.16) may be performed. A sum rule is obtained:

$$\sum_f \frac{d\sigma_{\text{tot}}^{(\text{inel})}}{d\Omega} = \frac{m^2}{(2\pi)^2 \hbar^4} \left\langle \frac{\mathbf{k}_f}{\mathbf{k}_i} | t(\mathbf{k}_f, \mathbf{k}_i) |^2 \right\rangle_{\text{av}, n, m} \sum (\langle \Psi_i | e^{i\mathbf{q} \cdot (\mathbf{r}_m - \mathbf{r}_n)} \Psi_i \rangle - \langle \Psi_i | e^{i\mathbf{q} \cdot \mathbf{r}_m} \Psi_i \rangle \langle \Psi_i | e^{-i\mathbf{q} \cdot \mathbf{r}_n} \Psi_i \rangle)$$

Using the antisymmetry of the wave function  $\Psi_i$ , the sum is readily shown to equal

$$\sum_{n, m} (\langle \Psi_i | e^{i\mathbf{q} \cdot (\mathbf{r}_m - \mathbf{r}_n)} \Psi_i \rangle - \langle \Psi_i | e^{i\mathbf{q} \cdot \mathbf{r}_m} \Psi_i \rangle \langle \Psi_i | e^{-i\mathbf{q} \cdot \mathbf{r}_n} \Psi_i \rangle) = A + A(A-1)\tilde{\rho}(\mathbf{q}, -\mathbf{q}) - A^2\tilde{\rho}(\mathbf{q})\tilde{\rho}(-\mathbf{q}) \quad (2.17)$$

where

$$\tilde{\rho}(\mathbf{q}, -\mathbf{q}) = \int e^{i\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} \rho(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad (2.18)$$

and

$$\rho(\mathbf{r}_1, \mathbf{r}_2) \equiv \int |\Psi_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots)|^2 d\mathbf{r}_3 \dots \quad (2.19)$$

The quantity  $\rho(\mathbf{r}_1, \mathbf{r}_2)$  is the diagonal two-body density giving the probability density to find a nucleon in the range  $d\mathbf{r}_1$  at  $\mathbf{r}_1$  and another in the range  $d\mathbf{r}_2$  at  $\mathbf{r}_2$ . To obtain the correlation density, one must subtract the probability that obtains when the two particles are independent,  $\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)$ . Thus

$$C(\mathbf{r}_1, \mathbf{r}_2) = \rho(\mathbf{r}_1, \mathbf{r}_2) - \rho(\mathbf{r}_1)\rho(\mathbf{r}_2) \quad (2.20)$$

Note that

$$\int C(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 = 0 = \int C(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_2 \quad (2.21)$$

The Fourier transform of  $C$  is

$$\tilde{C}(\mathbf{q}_1, \mathbf{q}_2) = \int e^{i\mathbf{q}_1 \cdot \mathbf{r}_1 + i\mathbf{q}_2 \cdot \mathbf{r}_2} C(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad (2.22)$$

From (2.21)

$$\tilde{C}(\mathbf{q}_1, 0) = \tilde{C}(0, \mathbf{q}_2) = 0 \quad (2.23)$$

Asymptotically,  $C(\mathbf{r}_1, \mathbf{r}_2)$  should vanish since for large  $|\mathbf{r}_1 - \mathbf{r}_2|$ , the nucleons are expected to be independent, so that

$$C(\mathbf{r}_1, \mathbf{r}_2) \rightarrow 0 \quad \text{as } |\mathbf{r}_1 - \mathbf{r}_2| \rightarrow \infty \quad (2.24)$$

In terms of the function  $C$ , the right-hand side of (2.17) is

$$A + A(A - 1)\tilde{C}(\mathbf{q}, -\mathbf{q}) - A|\rho(\mathbf{q})|^2$$

so that

$$\sum_f \frac{d\sigma_{\text{tot}}^{(\text{inel})}}{d\Omega} = \frac{m^2}{(2\pi)^2 \hbar^4} \left\langle \frac{k_f}{k_i} |t(\mathbf{k}_f, \mathbf{k}_i)|^2 \right\rangle_{\text{av}} [A(1 - |\rho(\mathbf{q})|^2) + A(A - 1)\tilde{C}(\mathbf{q}, -\mathbf{q})] \quad (2.25)$$

We see immediately the expected result that this cross section vanishes as  $q \rightarrow 0$ . As already indicated [see (1.4)] and to be shown in more detail below,  $\tilde{C} \sim 1/A$ , so that  $\langle d\sigma_{\text{tot}}^{(\text{inel})}/d\Omega \rangle_{\text{av}} \sim A$ . This result is to be contrasted with the elastic scattering cross section, which according to (2.7), is proportional to  $A^2$ . The latter is a consequence of the constructive interference of the waves scattered by the target nucleon in the forward direction. For this reason, the elastic scattering is referred to as *coherent scattering*. The inelastic cross section is referred to as *incoherent scattering* since proportionality of the cross section to  $A$  can be interpreted as addition of the cross sections, rather than the amplitude, for each target nucleon.

An important consequence of this discussion of inelastic scattering is that it will be easier to observe the correlation term at the larger angles,  $\mathbf{q}R \sim 1$ , and in the inelastic scattering.

A simple model of  $\rho(\mathbf{r}_1, \mathbf{r}_2)$  will serve to illustrate some of these points. We take

$$\rho(\mathbf{r}_1, \mathbf{r}_2) = Ne^{-(1/2)\alpha^2(r_1^2 + r_2^2)} [1 - e^{-(1/2)\beta^2(r_1 - r_2)^2}] \quad (2.26)$$

The correlation is carried by the second term in parentheses. It disappears as  $|\mathbf{r}_1 - \mathbf{r}_2| \rightarrow \infty$ . Moreover,  $\rho(\mathbf{r}_1, \mathbf{r}_2)$  goes to zero as  $|\mathbf{r}_1 - \mathbf{r}_2| \rightarrow 0$ , simulating the effect of a hard core and/or the Pauli exclusion principle. The parameter  $\alpha$  is  $O(1/R)$  and  $\beta$  is  $O(1/r_c)$ , where  $r_c$  is the correlation length, so that

$$\beta \gg \alpha \quad (2.27)$$

except for the smallest nuclei.  $N$  is a normalization factor determined by the condition

$$\iint \rho(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 = 1$$

Note that

$$\rho(\mathbf{r}_1) = \int \rho(\mathbf{r}_1, \mathbf{r}_2) d\mathbf{r}_2$$

has a contribution from the correlation term in  $\beta$ :

$$\begin{aligned} \rho(\mathbf{r}) &= N \frac{(2\pi)^{3/2}}{\alpha^3} \left[ e^{-(1/2)\alpha^2 r^2} - \left( \frac{\alpha^2}{\alpha^2 + \beta^2} \right)^{3/2} e^{-(1/2)[(\alpha^2 + 2\beta^2)/(\alpha^2 + \beta^2)]\alpha^2 r^2} \right] \\ &\xrightarrow{\beta \gg \alpha} N \frac{(2\pi)^{3/2}}{\alpha^3} \left[ e^{-(1/2)\alpha^2 r^2} - \left( \frac{\alpha}{\beta} \right)^3 e^{-\alpha^2 r^2} \right] \end{aligned} \quad (2.28)$$

The second term in brackets has a shorter range than the first term. The normalization factor is

$$N = \frac{\alpha^6}{(2\pi)^3 \{1 - [\alpha^2/(\alpha^2 + \beta^2)]^{3/2}\}} \xrightarrow{\beta \gg \alpha} \frac{\alpha^6}{(2\pi)^3 [1 - (1/\sqrt{8})(\alpha^3/\beta^3)]} \quad (2.29)$$

It is straightforward to evaluate the Fourier transforms:

$$\begin{aligned} \tilde{\rho}(\mathbf{q}) &= \frac{1}{1 - [\alpha^2/(\alpha^2 + 2\beta^2)]^{3/2}} \left\{ e^{-q^2/2\alpha^2} - \left( \frac{\alpha^2}{\alpha^2 + 2\beta^2} \right)^{3/2} e^{-(1/2)q^2[\alpha^2 + \beta^2]/\alpha^2(\alpha^2 + 2\beta^2)} \right\} \\ &\xrightarrow{\beta \gg \alpha} \frac{1}{1 - (1/\sqrt{8})(\alpha^3/\beta^3)} \left( e^{-q^2/2\alpha^2} - \frac{1}{\sqrt{8}} \frac{\alpha^3}{\beta^3} e^{-q^2/4\alpha^2} \right) \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} \tilde{\rho}(\mathbf{q}_1, \mathbf{q}_2) &= \frac{1}{1 - [\alpha^2/(\alpha^2 + 2\beta^2)]^{3/2}} \left\{ e^{-(q_1^2 + q_2^2)/2\alpha^2} \right. \\ &\quad \left. - \left( \frac{\alpha^2}{\alpha^2 + 2\beta^2} \right)^{3/2} e^{-(1/2)[1/\alpha^2(\alpha^2 + 2\beta^2)][\alpha^2(q_1^2 + q_2^2) + \beta^2(\mathbf{q}_1 + \mathbf{q}_2)^2]} \right\} \end{aligned} \quad (2.31)$$

Finally,

$$\begin{aligned} \tilde{C}(\mathbf{q}_1, \mathbf{q}_2) &\xrightarrow{\beta \gg \alpha} -\frac{\alpha^3}{\sqrt{8}\beta^3} \left[ e^{-(1/2)(q_1^2 + q_2^2)/\alpha^2} + e^{-(1/4)\alpha^2\beta^2[\alpha^2(q_1^2 + q_2^2) + \beta^2(\mathbf{q}_1 + \mathbf{q}_2)^2]} \right. \\ &\quad \left. - (e^{-q_1^2/4\alpha^2 - q_2^2/2\alpha^2} + e^{-q_1^2/2\alpha^2 - q_2^2/4\alpha^2}) \right] \end{aligned} \quad (2.32)$$

We see directly that for  $\beta \gg \alpha$ ,  $\tilde{C}$  is on the order of  $(r_c/R)^3$ , so that  $\tilde{C} \sim 1/A$ .  $\tilde{C}(\mathbf{q}_1, \mathbf{q}_2)$  goes to zero as  $\mathbf{q}_1$  or  $\mathbf{q}_2$  go to zero. Note also that for large  $\mathbf{q}_1$  and

$\mathbf{q}_2$  the decrease of  $\tilde{C}$  is governed by the parameters  $1/\alpha^2$ ,  $1/2\alpha^2$  and when  $\mathbf{q}_1 + \mathbf{q}_2 = 0$ , by  $1/2\beta^2$ . This last case will provide the smallest asymptotic rate of decrease since  $1/\beta^2 \ll 1/\alpha^2$ . The sum rule, (2.25), depends on  $\tilde{C}(\mathbf{q}, -\mathbf{q})$ :

$$\tilde{C}(\mathbf{q}, -\mathbf{q}) \xrightarrow{\beta \gg \alpha} -\frac{\alpha^3}{\sqrt{8}\beta^3} [e^{-q^2/\alpha^2} + e^{-q^2/2\beta^2} - 2e^{-(3/4)(q^2/\alpha^2)}], \quad (2.33)$$

which at large angles is dominated by the second term.

### 3. OPTICAL MODEL POTENTIAL<sup>‡</sup>

The preceding discussion assumes that the projectile wave incident on each target nucleon is the incident plane wave. In fact, the incident wave is composed of that plane wave together with the waves that have been generated by scattering from all the other nucleons of the target nucleus. The discussion in Section 2 is thus invalid if the probability for secondary and multiple scattering, that is, rescattering of wave generated by a previous scattering, is important. When that is the case, the wave incident on a target nucleon consists of a linear superposition of plane wave so that (2.1) is replaced by

$$\mathcal{F}_{e1} \equiv \langle \Psi_i | \hat{\mathcal{F}} \Psi_i \rangle = \sum_n \int \frac{d\mathbf{k}}{(2\pi)^3} \langle \Psi_i | t(\mathbf{k}_f, \mathbf{k}; \mathbf{r}_n) \Psi_i \rangle \tilde{\chi}_n(\mathbf{k})$$

or

$$\mathcal{F}_{e1} = A \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{r}_n \rho(\mathbf{r}_n) t(\mathbf{k}_f, \mathbf{k}; \mathbf{r}_n) \tilde{\chi}_n(\mathbf{k}) \quad (3.1)$$

where  $\tilde{\chi}_n(\mathbf{k})$  is the probability amplitude for a plane wave of momentum  $\mathbf{k}$  to be incident on the  $n$ th target nucleon. We now introduce  $t(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n)$  by inverting the Fourier transform, (2.5a), and using (2.5b) as well, so that

$$\mathcal{F}_{e1} = A \int \frac{d\mathbf{k}}{(2\pi)^3} \int \rho(\mathbf{r}_n) d\mathbf{r}_n \int d\mathbf{r} \int d\mathbf{r}' e^{-i\mathbf{k}_f \cdot \mathbf{r}} t(\mathbf{r} - \mathbf{r}_n, \mathbf{r}' - \mathbf{r}_n) e^{i\mathbf{k} \cdot \mathbf{r}'} \tilde{\chi}_n(\mathbf{k}) \quad (3.2)$$

The integral over  $\mathbf{k}$  yields directly the incident wave in coordinate space:

$$\chi_n(\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{r}'} \tilde{\chi}_n(\mathbf{k}) \quad (3.3)$$

<sup>‡</sup>Lax (51).

so that (3.2) can be rewritten

$$\mathcal{F}_{e1} = A \int d\mathbf{r} e^{-i\mathbf{k}_f \cdot \mathbf{r}} \int d\mathbf{r}' \left[ \int d\mathbf{r}_n t(\mathbf{r} - \mathbf{r}_n, \mathbf{r}' - \mathbf{r}_n) \rho(\mathbf{r}_n) \right] \chi_n(\mathbf{r}') \quad (3.4)$$

We now make the approximation that  $\chi_n(\mathbf{r}')$  is independent of  $n$  and can be written  $\chi(\mathbf{r}')$ . This is not exact since the linear combination of waves incident on the  $n$ th nucleon, (3.1), should not include the effect of the wave coming from the  $n$ th nucleon itself. However, if the number of nucleons is sufficiently large, the error should be small. When  $\chi_n$  is replaced by  $\chi$ , the resulting amplitude is identical to that which would be obtained from a Schrödinger equation with the nonlocal energy-dependent optical model potential  $V^{(\text{opt})}(\mathbf{r}, \mathbf{r}')$ :

$$V^{(\text{opt})}(\mathbf{r}, \mathbf{r}') = A \int d\mathbf{r}_n t(\mathbf{r} - \mathbf{r}_n, \mathbf{r}' - \mathbf{r}_n) \rho(\mathbf{r}_n) \quad (3.5)$$

Its Fourier transform is

$$\begin{aligned} \tilde{V}^{(\text{opt})}(\mathbf{k}, \mathbf{k}') &= \iint d\mathbf{r} d\mathbf{r}' e^{-i\mathbf{k} \cdot \mathbf{r}} V^{(\text{opt})}(\mathbf{r}, \mathbf{r}') e^{i\mathbf{k}' \cdot \mathbf{r}'} \\ &= A \tilde{\rho}(\mathbf{k} - \mathbf{k}') \tilde{t}(\mathbf{k}, \mathbf{k}') \end{aligned} \quad (3.6)$$

In using (3.6) it has been the practice to structure  $\tilde{t}(\mathbf{k}_f, \mathbf{k}_i)$  as follows:

$$\tilde{t}(\mathbf{k}_f, \mathbf{k}_i) = t_E(\mathbf{q}) \quad \mathbf{q} = \mathbf{k}_i - \mathbf{k}_f \quad (3.7)$$

where  $E$ , the projectile energy, is treated as a parameter. Under these circumstances

$$\begin{aligned} t(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n) &= \delta(\mathbf{r} - \mathbf{r}') \frac{1}{(2\pi)^3} \int e^{-i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_n)} \tilde{t}_E(\mathbf{q}) d\mathbf{q} \\ &\equiv \delta(\mathbf{r} - \mathbf{r}') t_E(\mathbf{r} - \mathbf{r}_n) \end{aligned} \quad (3.7')$$

The optical potential then becomes local:

$$\begin{aligned} V^{(\text{opt})}(\mathbf{r}', \mathbf{r}) &= \delta(\mathbf{r} - \mathbf{r}') V_E^{(\text{opt})}(\mathbf{r}) \\ V_E^{(\text{opt})}(\mathbf{r}) &= A \int d\mathbf{r}_n t_E(\mathbf{r} - \mathbf{r}_n) \rho(\mathbf{r}_n) \end{aligned} \quad (3.8)$$

This is the form of the high-energy multiple scattering optical potential found in the literature [Lax (51), Kerman, McManus, and Thaler (59)]. Its Fourier transform is

$$\tilde{V}_E^{(\text{opt})}(\mathbf{q}) = \int e^{i\mathbf{q} \cdot \mathbf{r}} V_E^{(\text{opt})}(\mathbf{r}) = A \tilde{t}_E(\mathbf{q}) \tilde{\rho}(\mathbf{q}) \quad (3.9)$$

Both (3.5) and (3.8) demonstrate that within the limits of the approximations employed to obtain them, that high-energy elastic scattering experiments provide a probe, symbolized by the transition matrix  $t(\mathbf{k}_f, \mathbf{k}_i)$  characteristic of the projectile–nucleon scattering, to study the one-body density  $\rho(r)$ . Each projectile, whether a nucleon,  $\alpha$ -particle, pion, and so on, will be sensitive to different aspects (i.e., spin and isospin dependence) of  $\rho$  so that by combining experiments one may be able to obtain a complete description of  $\rho$ .

Even in the approximate form, (3.5) and (3.8), the transition amplitude for all values of  $\mathbf{k}_f$  and  $\mathbf{k}_i$  needed to obtain  $V^{(\text{opt})}$  cannot be determined directly from experiment since one would have to know  $\tilde{t}(\mathbf{k}_f, \mathbf{k}_i)$  for  $k_f \neq k_i$ . The procedure generally used takes a functional form for  $\tilde{t}_E(\mathbf{q})$  fitted to experimental data and using that form extrapolates to values “off the energy shell” (i.e.,  $k_f \neq k_i$ ). A commonly used form is

$$\tilde{t}_E(\mathbf{q}) = B(E)e^{-b(E)q^2} \quad (3.10)$$

This procedure leads to minor errors, for the following reason. For larger nuclei, only values of  $\tilde{t}_E(\mathbf{q})$  near  $q = 0$  will enter importantly into the optical potential. The reasons follow. The scale of  $\tilde{\rho}(\mathbf{q})$  is  $1/R$  ( $R = \text{nuclear radius}$ ). For infinite nuclei  $\tilde{\rho}(\mathbf{q})$  is proportional to  $\delta(\mathbf{q})$ , while the scale of  $\tilde{t}$  is  $1/r_N$ , where  $r_N$  is the range of nuclear forces. Hence  $\tilde{t}(\mathbf{q})$  falls off with increasing  $q$  much more slowly than  $\tilde{\rho}(\mathbf{q})$ . In the relatively small momentum transfer range in which  $\tilde{\rho}$  differs from zero, there is no difficulty in obtaining the requisite  $\tilde{t}(\mathbf{q})$  from experiment.

For very large nuclei (i.e., nuclear matter), (3.9) becomes

$$\tilde{V}_E^{(\text{opt})}(\mathbf{q}) = A\tilde{t}_E(0)\tilde{\rho}(\mathbf{q})$$

and

$$V_E^{(\text{opt})}(\mathbf{r}) = A\tilde{t}_E(0)\rho(\mathbf{r}) \quad (3.11)$$

The imaginary part of  $V_E^{(\text{opt})}$  can be related to  $\lambda$ , the mean free path of the projectile in nuclear matter. We note that

$$\frac{2m}{\hbar^2} \text{Im } V_E^{(\text{opt})} = \frac{2m}{\hbar^2} A\rho \text{Im } \tilde{t}_E(0) = -4\pi A\rho \text{Im } f(0^\circ) = -A\rho k\sigma_T \quad (3.12)$$

where in the last step we have made use of the relationship between the imaginary part of the elastic scattering amplitude at  $0^\circ$ ,  $f(0^\circ)$ , and the total cross section:

$$\sigma_T = \frac{4\pi}{k} \text{Im } f(0^\circ) \quad (3.13)$$

From the Schrödinger equation one has

$$(k_R + ik_I)^2 = \frac{2m}{\hbar^2} E - \frac{2m}{\hbar^2} V^{(\text{opt})} \quad (3.14)$$

where  $k_R$  and  $k_I$  are the real and imaginary part of  $k$ , respectively. Inserting (3.11), using (3.12), and assuming that  $k_R \gg k_I$  leads to

$$k_R^2 = \frac{2m}{\hbar^2} (E - V_R^{(\text{opt})})$$

and

$$\frac{2m}{\hbar^2} V_I^{(\text{opt})} = -2k_R k_I \simeq -2k k_I$$

Hence, using (3.12),

$$k_I = \frac{1}{2} A \rho \sigma_T \quad (3.15)$$

and the mean free path,  $\lambda$ ,

$$\lambda = \frac{1}{2k_I} = \frac{1}{A \rho \sigma_T} \quad (3.16)$$

a familiar result. (See Chapter V, p. 354, for a derivation that takes into account the important effect of the properties of the medium in which the collisions occur.)

The optical potential bears a simple relation to the results obtained in Section 2 for the scattering amplitude [see (2.7)]. Equation (2.7) is the first Born approximation amplitude using the optical model potentials, (3.5) or (3.8). The Schrödinger equation with these potentials takes into account the distortion of the incident plane wave by the nucleon medium. The Schrödinger equation also develops an amplitude that satisfies unitarity, which is not the case for the first Born approximation. Form equation (3.8) [rather than (3.5)] is often pointed to as justification for the *folding potential* described in Chapter V, to which the reader can turn for further discussion.

The foregoing derivation of the optical model potential fails under two circumstances. First, it fails if three-body forces are important [Austern (83)], for then the scattering cannot involve one nucleon at a time with the projectile on the energy shell between collisions. It also fails, even when only two-body forces are acting, when the target nucleons are too close to each other. In that event, a double scattering in which the projectile is not on the energy shell after the first scattering becomes possible, in contradiction to the postulated

conditions for the validity of (3.5) and (3.9). The importance of this process depends on the probability that the target nucleons are sufficiently close to each other, that is on the correlation function. If the energy deficiency is  $\Delta E$ , the lifetime of the system is  $\hbar/\Delta E$  and the distance traveled by the projectile is  $\hbar v/\Delta E$ . By the end of this time interval a second collision restoring the system to the energy shell is necessary, so that  $\hbar v/\Delta E$  must be on the order of the correlation length  $r_c$ . As  $\Delta E$  increases, the correlation length to which the experiment is sensitive decreases, so that the projectile must go farther off the energy shell to see smaller correlation lengths.

One can take account of these collisions in which two of the target nucleons are close together by considering multiple scattering as involving a series of scattering from two target nucleons rather than from one target nucleon as assumed in the earlier discussion. It is intuitively clear that in that case (3.8)<sup>†</sup> is replaced by

$$V^{(\text{opt})}(\mathbf{r}, \mathbf{r}') = A(A-1) \int d\mathbf{r}_m \int d\mathbf{r}_n t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n, \mathbf{r}_m) \rho(\mathbf{r}_n, \mathbf{r}_m) \quad (3.17)$$

In this equation  $t^{(2)}$  is the transition amplitude for the scattering of the projectile by two target nucleons located at  $\mathbf{r}_n$  and  $\mathbf{r}_m$ . It is generally a nonlocal operator. [It is left to the reader as a problem to derive (3.17) using an analysis following that which led to (3.8).] The two-particle density  $\rho(\mathbf{r}_n, \mathbf{r}_m)$  can, according to (2.20), be written as follows:

$$\rho(\mathbf{r}_n, \mathbf{r}_m) = \rho(\mathbf{r}_n)\rho(\mathbf{r}_m) + C(\mathbf{r}_n, \mathbf{r}_m) \quad (2.20)$$

where [cf. (2.21)]

$$\int C(\mathbf{r}_n, \mathbf{r}_m) d\mathbf{r}_n = 0 = \int C(\mathbf{r}_n, \mathbf{r}_m) d\mathbf{r}_m \quad (2.21)$$

Consider the term generated when the first term of (2.20) is inserted into (3.17):

$$A(A-1) \int d\mathbf{r}_m \int d\mathbf{r}_n \rho(\mathbf{r}_n) t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n, \mathbf{r}_m) \rho(\mathbf{r}_m)$$

This term describes the independent scattering of the projectile by nucleons at  $\mathbf{r}_n$  and  $\mathbf{r}_m$ . This is precisely what the optical potentials (3.5) and (3.8) were designed to describe. Hence this term is already contained in (3.8). The term arising from the correlation in (2.20) is

$$A(A-1) \int d\mathbf{r}_n \int d\mathbf{r}_m C(\mathbf{r}_n, \mathbf{r}_m) t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n, \mathbf{r}_m) \quad (3.18)$$

<sup>†</sup>The more complex equation, (3.5), could equally well be generalized.

Using (3.8) to describe the scattering from the two-particle system, one obtains

$$t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n, \mathbf{r}_m) = \frac{1}{2}t(\mathbf{r} - \mathbf{r}_n)\delta(\mathbf{r} - \mathbf{r}') + \frac{1}{2}t(\mathbf{r} - \mathbf{r}_m)\delta(\mathbf{r} - \mathbf{r}') \\ + \Delta t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n, \mathbf{r}_m) \quad (3.19)$$

The first two terms are obtained from (3.8); the last is the nonadditive contribution obtained when the scattering from the two-body system is obtained more precisely. Inserting into (3.18) and using (2.21), (3.18) becomes

$$A(A-1) \int d\mathbf{r}_n \int d\mathbf{r}_m C(\mathbf{r}_n, \mathbf{r}_m) \Delta t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_n, \mathbf{r}_m)$$

The revised optical potential taking into account the possibility that two of target nucleons can be close together is

$$V^{(\text{opt})}(\mathbf{r}, \mathbf{r}') = A \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t(\mathbf{r} - \mathbf{r}_1) \delta(\mathbf{r} - \mathbf{r}') \\ + A(A-1) \int d\mathbf{r}_1 \int d\mathbf{r}_2 C(\mathbf{r}_1, \mathbf{r}_2) \Delta t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_1, \mathbf{r}_2) \quad (3.20)$$

It is generally not possible to obtain  $\Delta t^{(2)}$  for the necessary ranges of  $\mathbf{r}_1$  and  $\mathbf{r}_2$  from experiment. Therefore, to complete (3.20), a method for calculating  $\Delta t^{(2)}$  must be given. One might numerically solve the problem of projectile/two-nucleon scattering in the frozen nucleus approximation [Sparrow (75)] or one can provide an approximation [Feshbach, Gal, and Hüfner (71); Chaumeaux, Layly, and Schaeffer (78)], which permits a ready evaluation of this term. By taking the Fourier transforms of the  $C \Delta t^{(2)}$  term, one can verify that because of the properties of  $\tilde{C}(\mathbf{q}_1, \mathbf{q}_2)$  [see (2.32)], the magnitude of that term is on the order of  $(r_c/R)^3$  ( $\sim 1/A$ ) multiplied by the magnitude of  $\Delta t^{(2)}(\mathbf{q}_1, \mathbf{q}_2)$ . The first factor reduces the  $A$  dependence of  $V^{(\text{opt})}$  to a linear one. The second is expected to be small at high energies since most of the scattering by the two-body system will be given by the first two terms of (3.19). One therefore expects the second term in (3.20) to be small compared to the first term (see Appendix B of this chapter). However, because of interference of the first and second term, the latter may become visible especially at high momentum transfers, where the precipitous decrease of the first term will be modified by the presence of the second.

One should note the differing origin of the correlation function present in (2.25) and (3.20). In (2.25) we are concerned with the inelastic cross section. There is no interference and only  $\tilde{C}(\mathbf{q}, -\mathbf{q})$  makes its appearance. Only single scattering is present in the amplitude,  $\tilde{C}$  making its appearance as a consequence of squaring the transition amplitude and summing over all final states. The conclusion reached in Section 2, that  $\tilde{C}$  will not be visible in the elastic scattering,

must be modified because as has been emphasized, interference of the first and second terms in (3.20) may make the correlation-dependent term visible.

#### 4. FORMAL THEORY OF MULTIPLE SCATTERING<sup>‡</sup>

The intuitive considerations of the preceding sections need to be put on a firmer footing in which a more accurate result is derived, with the approximations clearly stated and methods for the calculation of corrections indicated. In this section we employ the formalism developed by Kerman, McManus, and Thaler (59) [here called the KMT method; see also Feshbach, Gal, and Hüfner (71)], which in turn is based on the analysis by Watson (53, 57, 58) and Lax (51) of the multiple scattering problem.

Formally, the multiple scattering problem can be stated as follows. Let the potential acting between the incident projectile and the target nucleus be a sum of two-body interactions,  $v_i$ , including spin and isospin dependence acting between the projectile and the  $i$ th target nucleon:

$$V = \sum_{i=1}^A v_i \mathcal{A} \quad (4.1)$$

where  $\mathcal{A}$  is the antisymmetrization operator operating on the target nucleons, thus guaranteeing that only those wave functions for the target system that satisfy the Pauli principle will enter into the discussion. When the projectile consists of nucleons, we shall assume that the Pauli principle acting between the projectile nucleons and the target nucleons need not be enforced [Takeda and Watson (55)]. Physically, this seems reasonable (unless the collision leads to a large energy loss), for one can identify the projectile after collision by its large energy. However, there have been some criticisms of this procedure [Picklesimer and Thaler (81)].

The goal of the multiple scattering theory is to relate the transition matrix,  $\mathcal{F}$ , for the projectile–nucleus collision to the transition matrix,  $t$ , for the projectile–nucleon collision. From (4.1),  $\mathcal{F}$  satisfies

$$\mathcal{F}(E) = \sum_{i=1}^A v_i \mathcal{A} + \sum_{i=1}^A v_i \frac{\mathcal{A}}{\alpha} \mathcal{F}(E) \quad (4.2)$$

where

$$\alpha = E^{(+)} - K - H_N \quad (4.3)$$

and  $H_N$  is the target nucleus Hamiltonian;  $K$  is the kinetic energy of the incident projectile relative to the center of mass of the target nucleus. On the other hand,

<sup>‡</sup>Kerman, McManus, and Thaler (59); Feshbach, Gal, and Hüfner (71); Feshbach (81).

the  $t$  matrix satisfies, for scattering from the  $i$ th nucleon,

$$t_i(E) = v_i + v_i \frac{1}{E^{(+)} - K_0} t_i(E) \quad (4.4)$$

where  $K_0$  is the kinetic energy operator in the projectile–nucleon system.

As a first step, one introduces an operator  $\tau$  which is the transition matrix for the averaged two-body interaction  $(1/A)\sum_1^A v_i \mathcal{A}$ :

$$\tau = \frac{1}{A} \sum_1^A v_i \mathcal{A} + \frac{1}{A} \sum_1^A v_i \frac{\mathcal{A}}{\alpha} \tau \quad (4.5)$$

$\tau$  is a many-body operator closely related to  $t_i$ . This relationship is made more explicit by introducing  $\tau_i$  defined by

$$\tau = \frac{1}{A} \sum \tau_i \quad (4.6)$$

Then

$$\tau_i = v_i \mathcal{A} + v_i \frac{\mathcal{A}}{\alpha} \tau \quad (4.7)$$

which should be compared with (4.4).  $\tau_i$  may be considered as the effective two-body operator in the nuclear medium. The latter's presence is indicated by the antisymmetrization operator  $\mathcal{A}$  as well as by the nuclear Hamiltonian in the operator  $\alpha^{-1}$ . Equation (4.7) takes into account the contribution to the scattering amplitude generated by the  $i$ th nucleon of the waves emanating from all the other nucleons, as indicated by the presence of the operator  $\tau$  on the right-hand side.

We can now use (4.5) to eliminate  $v_i$  from (4.2) for  $\tau$ . This elimination is essential if  $v_i$  is singular. Toward that end, rewrite (4.5) as follows:

$$\sum_i v_i \mathcal{A} = \left( A - \sum_i v_i \frac{\mathcal{A}}{\alpha} \right) \tau$$

and replace  $\sum v_i \mathcal{A}$  in (4.2) by the right-hand side:

$$\mathcal{F} = \left( A - \sum_i v_i \frac{\mathcal{A}}{\alpha} \right) \tau + \left( A - \sum_i v_i \frac{\mathcal{A}}{\alpha} \right) \tau \frac{1}{\alpha} \mathcal{F}$$

Using (4.5) [**Problem.** Prove (4.8)]

$$\tau \frac{1}{\alpha} \sum_i v_i \mathcal{A} = \sum_i v_i \frac{\mathcal{A}}{\alpha} \tau \quad (4.8)$$

this equation becomes

$$\mathcal{F} = A\tau + A\tau \frac{1}{\alpha} \mathcal{F} - \tau \frac{1}{\alpha} \left( \sum_i v_i \frac{\mathcal{A}}{\alpha} + \sum_i v_i \frac{\mathcal{A}}{\alpha} \frac{1}{\alpha} \mathcal{F} \right)$$

or

$$\mathcal{F} = A\tau + (A-1)\tau \frac{1}{\alpha} \mathcal{F} \quad (4.9)$$

Define  $\mathcal{F}'$  by

$$\mathcal{F} \equiv \frac{A}{A-1} \mathcal{F}' \quad (4.10)$$

Then

$$\mathcal{F}' = (A-1)\tau + (A-1)\tau \frac{1}{\alpha} \mathcal{F}' \quad (4.11)$$

We thus obtain the remarkable result that the scattering induced by  $\sum_i v_i$  can equally well be considered as a consequence of the effective interaction  $(A-1)\tau (= [(A-1)/A] \sum \tau_i)$ . The effect of the Pauli principle is now contained within the operator  $\tau$ , while the transition matrix  $\mathcal{F}$  is to be obtained from the solution of (4.11) by multiplication by the factor  $A/A-1$  according to (4.10).

With this result it now becomes possible to introduce the “frozen” nucleus approximation with some improvement upon its formulation as given in Section 2. We return to the Schrödinger equation equivalent to (4.11):

$$[E - K - H_N - (A-1)\tau]\Psi = 0 \quad (4.12)$$

and derive an equation for the open-channel component of  $\Psi$ . That component will at least contain the elastic channel, but it can as well contain other channels of interest. Toward this end we introduce a projection operator  $P$  which when applied to any wave function such as  $\Psi$  will yield the open-channel component of interest.  $P$  is given by

$$P = 0\rangle\langle 0 + 1\rangle\langle 1 + \dots \quad (4.13)$$

where  $0\rangle$  is the state vector for the ground state of the target nucleus,  $1\rangle$  the first excited state, and so on. The number of terms included is determined by the physics of the phenomena under study. Here the emphasis is on elastic and inelastic scattering. The projection operator  $Q$  complementary to  $P$  is defined by

$$Q \equiv 1 - P \quad (4.14)$$

The following relationships will be needed:

$$P^2 = P \quad Q^2 = Q \quad PQ = QP = 0 \quad (4.15)$$

We also define the symbols  $\tau_{PQ}$ ,  $\tau_{PP}$ , and  $\tau_{QQ}$ :

$$\begin{aligned} \tau_{PQ} &\equiv P\tau Q & \tau_{QP} &= Q\tau P \\ \tau_{PP} &\equiv P\tau P & \tau_{QQ} &= Q\tau Q \end{aligned}$$

The Schrödinger equation (4.12) can be replaced by a pair of coupled equations for  $P\Psi$  and  $Q\Psi$ , where  $P\Psi + Q\Psi = \Psi$ :

$$[E - K - (H_N)_{PP} - (A - 1)\tau_{PP}](P\Psi) = (A - 1)\tau_{PQ}(Q\Psi) \quad (4.16a)$$

$$[E - K - (H_N)_{QQ} - (A - 1)\tau_{QQ}](Q\Psi) = (A - 1)\tau_{QP}(P\Psi) \quad (4.16b)$$

Solving the second equation formally for  $(Q\Psi)$  and substituting in the first equation yields

$$\left[ E - K - (H_N)_{PP} - (A - 1)\tau_{PP} - (A - 1)\tau_{PQ} \frac{1}{E - K - (H_N)_{QQ} - (A - 1)\tau_{QQ}} (A - 1)\tau_{QP} \right] P\Psi = 0 \quad (4.17)$$

thereby deriving an effective Hamiltonian and in particular an effective potential for the subspace projected by  $P$ . Equation (4.17) is exact. The first-order term in the effective potential,  $(A - 1)\tau_{PP}$ , is supplemented by a second-order term involving  $\tau$  twice, which describes the system making the transition from the space projected by  $P$  to the complementary space projected by  $Q$ , propagating in  $\mathcal{Q}$  space followed by a transition back to the space projected by  $P$ . The Schrödinger equation iterates this process. Equation (4.17) is exact, but it is in a form that is suitable for approximation. For example, the frozen nucleus approximation is obtained by replacing  $(H_N)_{QQ}$  by an average excitation energy  $\bar{\epsilon}$  and  $(A - 1)\tau_{QQ}$  by a first approximation to the two-body projectile-nucleus optical model Hamiltonian. Hence

$$\begin{aligned} V_{\text{opt}} &\simeq (A - 1)\tau_{PP} + (A - 1)\tau_{PQ} \frac{1}{E - K - \bar{\epsilon} - V^{(1)}} (A - 1)\tau_{QP} \\ &= (A - 1)\tau_{PP} + (A - 1)^2 \tau_{PQ} \frac{Q}{E - K - \bar{\epsilon} - V^{(1)}} \tau_{QP} \end{aligned} \quad (4.17')$$

where  $V^{(1)}$  is the first-order potential [given by (4.30)] still to be derived. In writing the second of the equations we have used the fact the propagator no

longer depends on the target nuclear coordinates, depending only on the projectile coordinates relative to the nucleus center of mass. To proceed further we must elucidate the relation between  $\tau_i$  and  $t_i$ .

The first approximation replaces (4.4) by

$$t_i \simeq v_i + v_i \frac{1}{\alpha} t_i \quad (4.18)$$

This involves adding  $H_N$  to the denominator of the propagator to obtain  $1/\alpha$ . This may not be a serious error under the assumption that  $E$  is large. Since we shall eventually replace  $H_N$  by some average value, this error can be compensated to some extent by shifting the energy in (4.4). For a further discussion of this point, see Appendix A at the end of this chapter, where it is shown that the error is of the order of  $1/A$ . Note also that (4.4) is in the projectile–nucleon center-of-mass system, whereas (4.18) is in the projectile–nucleus frame, very close for the heavier nuclei to the laboratory frame.

Using (4.18), one can eliminate  $v_i$  in (4.7) for  $\tau_i$ . From (4.18),

$$v_i = \left(1 - v_i \frac{1}{\alpha}\right) t_i$$

Substituting in (4.7) gives

$$\begin{aligned} \tau_i &= \left(1 - v_i \frac{1}{\alpha}\right) t_i \mathcal{A} + \left(1 - v_i \frac{1}{\alpha}\right) t_i \frac{\mathcal{A}}{\alpha} \tau \\ &= t_i \mathcal{A} + t_i \frac{\mathcal{A}}{\alpha} \tau - t_i \frac{1}{\alpha} \left(v_i \mathcal{A} + v_i \frac{\mathcal{A}}{\alpha} \tau\right) \end{aligned}$$

where the equation

$$v_i \frac{1}{\alpha} t_i = t_i \frac{1}{\alpha} v_i$$

has been used. This equation follows from (4.18). Using (4.7), the equation for  $\tau_i$  becomes

$$\tau_i = t_i \mathcal{A} + t_i \frac{1}{\alpha} (\tau - \tau_i)$$

The equation for  $\tau$  is then

$$\begin{aligned} \tau &= \frac{1}{A} \sum t_i \mathcal{A} + \frac{1}{A} \sum t_i \frac{1}{\alpha} (\tau - \tau_i) \\ &= \frac{1}{A} \sum t_i \mathcal{A} + \frac{1}{A} \sum t_i \frac{\mathcal{A} - 1}{\alpha} \tau_i \end{aligned} \quad (4.19)$$

where the following relation has been used:

$$\tau = \mathcal{A} \tau_i \quad (4.20)$$

and it is assumed that the operators are acting on antisymmetrized wave functions. It is important to replace the propagator  $(1/\alpha)[=1/(E^{(+)} - K - H_N)]$  by  $1/\tilde{\alpha}$ , which takes the effect of the nuclear medium more completely in account:

$$\tilde{\alpha} = E^{(+)} - K - H_N - (A - 1)\tau \quad (4.21)$$

These two propagators are related by

$$\frac{1}{\alpha} = \frac{1}{\tilde{\alpha}} + \frac{1}{\alpha} - \frac{1}{\tilde{\alpha}} = \frac{1}{\tilde{\alpha}} + \frac{1}{\alpha}(\tilde{\alpha} - \alpha) \frac{1}{\tilde{\alpha}}$$

or

$$\frac{1}{\alpha} = \frac{1}{\tilde{\alpha}} - \frac{1}{\alpha}(A - 1)\tau \frac{1}{\tilde{\alpha}} \quad (4.22)$$

But we need  $(\mathcal{A} - 1)/\alpha$  and this is, from (4.22),

$$\frac{\mathcal{A} - 1}{\alpha} = \frac{\mathcal{A} - 1}{\tilde{\alpha}} \quad (4.23)$$

which follows from the equation  $(\mathcal{A} - 1)\mathcal{A} = 0$  and (4.20). Inserting this result in (4.19), we have

$$\tau = \frac{1}{A} \sum_i t_i \mathcal{A} + \frac{1}{A} \sum_i t_i \frac{\mathcal{A} - 1}{\tilde{\alpha}} \tau_i \quad (4.24)$$

providing the desired relation between  $\tau$  and  $t_i$ . [It is the analog of the Bethe–Goldstone equation stated in Chapter III of deShalit and Feshbach (74).] One can now solve this equation for  $\tau$  by successive approximations. In the first order, obtained by dropping the second term on the right-hand side of (4.24),

$$\tau \sim \frac{1}{A} \sum_i t_i$$

or

$$\tau_i \sim t_i \quad (4.25)$$

To second order obtained by using this result in (4.24),

$$\tau = \frac{1}{A} \sum_i t_i \mathcal{A} + \frac{1}{A} \sum_i t_i \frac{\mathcal{A} - 1}{\tilde{\alpha}} t_i \quad (4.26)$$

or

$$\tau = \frac{1}{A} \sum t_i + \frac{1}{A^2} \sum_{i,j} t_i \frac{1}{\tilde{\alpha}} t_j - \frac{1}{A} \sum t_i \frac{1}{\tilde{\alpha}} t_i \quad (4.26')$$

where we have used  $\mathcal{A}t_i = (1/A) \sum t_j$ . We can now calculate the effective potential operator  $\hat{V}_{\text{opt}}$ , the optical model potential  $\hat{V}_{\text{opt}} = P \hat{V}_{\text{opt}} P$ . From (4.17),

$$\begin{aligned} \hat{V}_{\text{opt}} &= (A-1)\tau + (A-1)^2 \tau \frac{Q}{\tilde{\alpha}} \\ &= (A-1)\tau + (A-1)^2 \tau \frac{1}{\tilde{\alpha}} - (A-1)^2 \tau \frac{P}{\tilde{\alpha}} \end{aligned}$$

where

$$\tilde{\alpha} = E - K - \bar{\epsilon} - V^{(1)} \quad (4.27)$$

One now inserts approximation (4.26') into  $\hat{V}_{\text{opt}}$ , making the additional approximation of replacing  $\tilde{\alpha}$  everywhere by  $\bar{\alpha}$ . The result is

$$\hat{V}_{\text{opt}} = \frac{A-1}{A} \sum t_i + \frac{A-1}{A} \sum_{i \neq j} t_i \frac{1}{\bar{\alpha}} t_j - \frac{(A-1)^2}{A^2} \sum_{i,j} t_i \frac{1}{\bar{\alpha}} t_j \quad (4.28)$$

This is the principal result of the multiple scattering formalism as developed by Kerman, McManus, and Thaler (59). Its extension to third order has been given by Ullo and Feshbach (74). There are three major approximations made in deriving (4.28). They are, in major part, high-energy approximations in that they become increasingly valid as the energy increases. That assertion depends in turn upon appropriate behaviour of the matrix elements of  $t_i$  and  $H_N$  with increasing momentum transfer,  $\mathbf{q}$ , and energy. What is required is that these matrix elements decrease rapidly enough with these increasing  $q$  and  $E$  so that the magnitude of  $t_i(1/\bar{\alpha})$  is sufficiently small to ensure convergence of the series for  $\hat{V}_{\text{opt}}$ , the first two orders of which are given by (4.28). This limitation can be avoided to a great extent if one were to solve the analog of the Bethe-Goldstone equation, (4.24) more exactly (i.e., adapting the independent pair approximation) rather than use a small perturbation approximation.

In the case of elastic scattering,  $P = 0 \rangle \langle 0$ , and

$$V_{\text{opt}} = \langle 0 | \hat{V}_{\text{opt}} | 0 \rangle \quad (4.29)$$

and  $(H_N)_{PP} = 0$ . The first-order optical model potential is thus

$$V_{\text{opt}}^{(1)} = \frac{A-1}{A} \langle 0 | \sum | 0 \rangle = (A-1) \langle 0 | t_1 | 0 \rangle \quad (4.30)$$

The transition matrix  $t_i$  is generally a nonlocal operator because of the presence of the second term in the Lippman–Schwinger equation, (4.4). In coordinate space it therefore has the form  $\langle \mathbf{r} - \mathbf{r}_1, \frac{1}{2}(\mathbf{r} + \mathbf{r}_1) | \hat{t} | \mathbf{r}' - \mathbf{r}'_1, \frac{1}{2}(\mathbf{r}' + \mathbf{r}'_1) \rangle$ , where, to be specific, we have used relative coordinate and center-of-mass coordinates for a *nucleon projectile*. The generalization to other projectiles is straightforward. Conservation of momentum determines the dependence on center-of-mass coordinates:

$$t_1 = \delta(\frac{1}{2}(\mathbf{r} + \mathbf{r}_1) - \frac{1}{2}(\mathbf{r}' + \mathbf{r}'_1)) t(\mathbf{r} - \mathbf{r}_1, \mathbf{r}' - \mathbf{r}'_1) \quad (4.31)$$

Then

$$\begin{aligned} V_{\text{opt}}^{(1)}(\mathbf{r}, \mathbf{r}') &= (A - 1) \int d\mathbf{r}_1 \int d\mathbf{r}'_1 \Psi^*(\mathbf{r}_1, \mathbf{r}_2, \dots) t_1 \Psi(\mathbf{r}'_1, \mathbf{r}_2, \dots) d\mathbf{r}_2 \dots \\ &= (A - 1) \int d\mathbf{r}_1 \int d\mathbf{r}'_1 K(\mathbf{r}_1, \mathbf{r}'_1) t(\mathbf{r} - \mathbf{r}_1, \mathbf{r}' - \mathbf{r}'_1) \delta(\frac{1}{2}(\mathbf{r} + \mathbf{r}_1) - \frac{1}{2}(\mathbf{r}' + \mathbf{r}'_1)) \end{aligned} \quad (4.32)$$

where  $K(\mathbf{r}_1, \mathbf{r}'_1)$  is the one-body density matrix for the ground-state target nucleus

$$K(\mathbf{r}, \mathbf{r}') = \int d\mathbf{r}_2 \dots \Psi^*(\mathbf{r}, \mathbf{r}_2, \dots) \Psi(\mathbf{r}', \mathbf{r}_2, \dots) \quad (4.33)$$

In momentum space (4.32) becomes

$$\tilde{V}_{\text{opt}}^{(1)}(\mathbf{k}, \mathbf{k}') = \int d\mathbf{r} \int d\mathbf{r}' e^{-i\mathbf{k} \cdot \mathbf{r}} V_{\text{opt}}^{(1)}(\mathbf{r}, \mathbf{r}') e^{i\mathbf{k}' \cdot \mathbf{r}'} \quad (4.34)$$

Introducing Fourier transforms for  $K$  and  $t$

$$K(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^6} \int ds \int ds' e^{i\mathbf{s} \cdot \mathbf{r} - i\mathbf{s}' \cdot \mathbf{r}'} \tilde{K}(\mathbf{s}, \mathbf{s}') \quad (4.35)$$

and

$$\tilde{t}(\boldsymbol{\kappa}, \boldsymbol{\kappa}') = \int d\boldsymbol{\rho} \int d\boldsymbol{\rho}' e^{-i\boldsymbol{\kappa} \cdot \boldsymbol{\rho} + i\boldsymbol{\kappa}' \cdot \boldsymbol{\rho}'} t(\boldsymbol{\rho}, \boldsymbol{\rho}') \quad (4.36)$$

Equation (4.34) becomes (use relative and center-of-mass integration variables)

$$\tilde{V}_{\text{opt}}^{(1)}(\mathbf{k}, \mathbf{k}') = \frac{A - 1}{(2\pi)^3} \int ds \int ds' \tilde{K}(\mathbf{s}, \mathbf{s}') \tilde{t}\left(\frac{\mathbf{s} + \mathbf{k}}{2}, \frac{\mathbf{s}' + \mathbf{k}'}{2}\right) \delta(\mathbf{s} - \mathbf{k} - (\mathbf{s}' - \mathbf{k}')) \quad (4.37)$$

The interpretation of this equation is instructive. The projectile brings in a

momentum of  $\mathbf{k}'$ , the target nucleon  $-s'$ . Upon scattering the projectile acquires a momentum of  $\mathbf{k}$ , the target nucleon,  $-s$ . The delta function ensures that the momentum transferred to the projectile,  $(\mathbf{k} - \mathbf{k}')$ , is balanced by the amount transferred to the target nucleon  $(s - s')$ . The  $\tilde{t}$  matrix gives the amplitude for a transition from relative momentum  $\frac{1}{2}(s' + \mathbf{k}')$  to  $\frac{1}{2}(s + \mathbf{k})$ . Equation (4.37) takes into account the motion (sometimes referred to as the Fermi motion) of the nucleons in the target nucleus.

As a final development we consider the consequences of the assumption that  $t(\mathbf{r}, \mathbf{r}')$  is local [see (3.7)],

$$t(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')t(\mathbf{r}) \quad (4.38)$$

upon the first-order potential, (4.32). The evaluation is straightforward. One finds that  $V_{\text{opt}}^{(1)}(\mathbf{r}, \mathbf{r}')$  is local:

$$V_{\text{opt}}^{(1)}(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')v_{\text{opt}}^{(1)}(\mathbf{r}) \quad (4.39)$$

where

$$v_{\text{opt}}^{(1)}(\mathbf{r}) = (A - 1) \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t(\mathbf{r} - \mathbf{r}_1) \quad (4.40)$$

In momentum space

$$\begin{aligned} \tilde{v}_{\text{opt}}^{(1)}(\mathbf{q}) &= \int e^{i\mathbf{q}\cdot\mathbf{r}} v_{\text{opt}}^{(1)}(\mathbf{r}) d\mathbf{r} \\ &= (A - 1) \tilde{\rho}(\mathbf{q}) \tilde{t}(\mathbf{q}) \end{aligned} \quad (4.40')$$

Equation (4.39) is in agreement to order  $1/A$  with the result obtained by intuitive arguments in Section 3 [see (3.9)]. In making this comparison, one should bear in mind that the scattering amplitude obtained using the potential equation (4.39) must be multiplied by the factor  $(A/A - 1)$ , according to (4.10), in order to obtain the full  $\mathcal{F}$  matrix. Thus the intuitive result and the first-order result just obtained will give the same amplitude in the Born approximation. Further discussion is postponed until the second-order term of (4.28) is evaluated.

We first express the second-order term in coordinate space assuming  $t_i$  to be a local operator [see (3.7)]:

$$\langle \mathbf{r} | t_i | \mathbf{r}' \rangle = \delta(\mathbf{r} - \mathbf{r}') t(\mathbf{r} - \mathbf{r}_i)$$

Then

$$\begin{aligned} V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}') &= \langle 0, \mathbf{r} | \hat{V}_{\text{opt}}^{(2)} | 0, \mathbf{r}' \rangle \\ &= (A - 1)^2 \int d\mathbf{r}'' \int d\mathbf{r}''' \left\{ \frac{1}{A(A - 1)} \langle 0 | \sum_{i \neq j} \delta(\mathbf{r} - \mathbf{r}'') t(\mathbf{r} - \mathbf{r}_i) \langle \mathbf{r}'' | \frac{1}{\alpha} | \mathbf{r}''' \rangle \right. \\ &\quad \times \delta(\mathbf{r}''' - \mathbf{r}') t(\mathbf{r}' - \mathbf{r}_j) | 0 \rangle \\ &\quad \left. - \frac{1}{A^2} \langle 0 | \sum_i \delta(\mathbf{r} - \mathbf{r}'') t(\mathbf{r} - \mathbf{r}_i) | 0 \rangle \langle \mathbf{r}'' | \frac{1}{\alpha} | \mathbf{r}''' \rangle \langle 0 | \sum_j \delta(\mathbf{r}''' - \mathbf{r}') t(\mathbf{r}' - \mathbf{r}_j) | 0 \rangle \right\} \end{aligned}$$

Using the antisymmetry of the target nuclear wave function and performing the integrations over  $\mathbf{r}''$  and  $\mathbf{r}'''$ , one obtains

$$\begin{aligned} V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}') &= (A-1)^2 \{ \langle 0 | t(\mathbf{r} - \mathbf{r}_1) t(\mathbf{r}' - \mathbf{r}_2) | 0 \rangle - \langle 0 | t(\mathbf{r} - \mathbf{r}_1) | 0 \rangle \langle 0 | t(\mathbf{r}' - \mathbf{r}_2) | 0 \rangle \\ &\quad \times \langle \mathbf{r} | \frac{1}{\tilde{\alpha}} | \mathbf{r}' \rangle \} \\ &= (A-1)^2 \int d\mathbf{r}_1 \int d\mathbf{r}_2 t(\mathbf{r} - \mathbf{r}_1) \langle \mathbf{r} | \frac{1}{\tilde{\alpha}} | \mathbf{r}' \rangle t(\mathbf{r}' - \mathbf{r}_2) [\rho(\mathbf{r}_1, \mathbf{r}_2) - \rho(\mathbf{r}_1)\rho(\mathbf{r}_2)] \end{aligned}$$

or

$$V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}') = (A-1)^2 \int d\mathbf{r}_1 \int d\mathbf{r}_2 t(\mathbf{r} - \mathbf{r}_1) \langle \mathbf{r} | \frac{1}{\tilde{\alpha}} | \mathbf{r}' \rangle t(\mathbf{r}' - \mathbf{r}_2) C(\mathbf{r}_1, \mathbf{r}_2) \quad (4.41)$$

Clearly,  $V_{\text{opt}}^{(2)}$  is a nonlocal energy-dependent potential involving a scattering by a target nucleon at  $\mathbf{r}_2$ , a propagation from  $\mathbf{r}'$  to  $\mathbf{r}$ , and a second scattering by a target nucleon at  $\mathbf{r}_1$ . The correlation function measures the probability of a target nucleon being present at  $\mathbf{r}_1$  and another at  $\mathbf{r}_2$ . Comparing (4.41) with the intuitive derivation of (3.20), one can identify the  $\Delta t^{(2)}$  of that equation (to order  $1/A$ ).

$$\Delta t^{(2)}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_1, \mathbf{r}_2) = t(\mathbf{r} - \mathbf{r}_1) \langle \mathbf{r} | \frac{1}{\tilde{\alpha}} | \mathbf{r}' \rangle t(\mathbf{r}' - \mathbf{r}_2) \quad (4.42)$$

We also restate (4.41) in momentum space, as this is the form in which one finds it in the literature:

$$\begin{aligned} \tilde{V}_{\text{opt}}^{(2)}(\mathbf{k}, \mathbf{k}') &= (A-1)^2 \int \frac{d\mathbf{k}''}{(2\pi)^3} \int \frac{d\mathbf{k}'''}{(2\pi)^3} [\tilde{t}(\mathbf{k}'' - \mathbf{k}) \langle \mathbf{k}'' | \frac{1}{\tilde{\alpha}} | \mathbf{k}''' \rangle \tilde{t}(\mathbf{k}' - \mathbf{k}''')] \\ &\quad \times \tilde{C}(\mathbf{k}'' - \mathbf{k}, \mathbf{k}' - \mathbf{k}''') \end{aligned} \quad (4.43)$$

where

$$\tilde{C}(\mathbf{q}, \mathbf{q}') = \int d\mathbf{r}_1 \int d\mathbf{r}_2 e^{i\mathbf{q} \cdot \mathbf{r}_1 + i\mathbf{q}' \cdot \mathbf{r}_2} C(\mathbf{r}_1, \mathbf{r}_2) \quad (4.44)$$

A posteriori, it appears that the multiple scattering series for the optical potential is an expansion in terms of correlation functions of increasing order. The first-order term depends on the density, the second in the pair correlation. Ullo (74) has evaluated the third-order term and has shown that to  $O(1/A)$ ,  $V_{\text{opt}}^{(3)}$  depends on the triple correlation function  $C^{(3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$  lending support to this surmise.

**Problem.** Show that the triple correlation function is given by

$$\begin{aligned} C^{(3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) &= \rho^{(3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - \rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2)\rho(\mathbf{r}_3) - \rho^{(2)}(\mathbf{r}_1, \mathbf{r}_3)\rho(\mathbf{r}_2) \\ &\quad - \rho^{(2)}(\mathbf{r}_2, \mathbf{r}_3)\rho(\mathbf{r}_1) + 2\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\rho(\mathbf{r}_3) \end{aligned}$$

Show that

$$\int C^{(3)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) d\mathbf{r}_1 = C(\mathbf{r}_2, \mathbf{r}_3)$$

The separation into density- and correlation-dependent contributions  $V_{\text{opt}}^{(1)}$  and  $V_{\text{opt}}^{(2)}$  must be modified when the spin dependence of the two-body transition operators is taken into account. Here it will suffice to give an example. The complete treatment is given in Lambert and Feshbach (73) and Parmentola and Feshbach (82). Suppose that the incident projectile is a nucleon and that

$$t(\mathbf{r} - \mathbf{r}_i) = t^{(0)}(\mathbf{r} - \mathbf{r}_i) + t^{(s)}(\mathbf{r} - \mathbf{r}_i) \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}_i$$

Then for a spin 0 target nucleus,

$$\begin{aligned} V^{(1)}(\mathbf{r}, \mathbf{r}') &= (A-1) \langle 0 | t(\mathbf{r} - \mathbf{r}_1) | 0 \rangle \delta(\mathbf{r} - \mathbf{r}') \\ &= (A-1) \langle 0 | t^{(0)}(\mathbf{r} - \mathbf{r}_1) | 0 \rangle \delta(\mathbf{r} - \mathbf{r}') \end{aligned}$$

so that  $t^{(s)}$  does not appear in  $V^{(1)}$ . On the other hand, the equation for  $V^{(2)}(\mathbf{r}, \mathbf{r}')$  becomes

$$\begin{aligned} V^{(2)}(\mathbf{r}, \mathbf{r}') &= (A-1)^2 \langle \mathbf{r} | \frac{1}{\alpha} | \mathbf{r}' \rangle \\ &\quad \times \{ \langle 0 | [t^{(0)}(\mathbf{r} - \mathbf{r}_1) + t^{(s)}(\mathbf{r} - \mathbf{r}_1) \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}_1] [t^{(0)}(\mathbf{r} - \mathbf{r}_2) + t^{(s)}(\mathbf{r}' - \mathbf{r}_2) \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}_2] | 0 \rangle \\ &\quad - \langle 0 | t^{(0)}(\mathbf{r} - \mathbf{r}_1) | 0 \rangle \langle 0 | t^{(0)}(\mathbf{r}' - \mathbf{r}_2) | 0 \rangle \} \\ &= (A-1)^2 \langle \mathbf{r} | \frac{1}{\alpha} | \mathbf{r}' \rangle \int d\mathbf{r}_1 \int d\mathbf{r}_2 \{ t^{(0)}(\mathbf{r} - \mathbf{r}_1) t^{(0)}(\mathbf{r}' - \mathbf{r}_2) C(\mathbf{r}_1, \mathbf{r}_2) \\ &\quad + \langle 0 | t^{(s)}(\mathbf{r} - \mathbf{r}_1) t^{(s)}(\mathbf{r}' - \mathbf{r}_2) \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 | 0 \rangle \} \end{aligned}$$

To avoid some Racah algebra, assume that the target wave function is a product of a spin and of a space-dependent factor. Hence the second term becomes

$$\begin{aligned} M &\equiv \int d\mathbf{r}_1 \int d\mathbf{r}_2 \langle 0 | t^{(s)}(\mathbf{r} - \mathbf{r}_1) t^{(s)}(\mathbf{r}' - \mathbf{r}_2) \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 | 0 \rangle \\ &= \int d\mathbf{r}_1 \int d\mathbf{r}_2 \rho(\mathbf{r}_1, \mathbf{r}_2) t^{(s)}(\mathbf{r} - \mathbf{r}_1) t^{(s)}(\mathbf{r}' - \mathbf{r}_2) \langle 0 | \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 | 0 \rangle \end{aligned}$$

and

$$\langle 0 | \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 | 0 \rangle = \frac{1}{A(A-1)} \sum_{i \neq j} \langle 0 | \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j | 0 \rangle$$

$$\begin{aligned}
 &= \frac{1}{A(A-1)} [\langle 0 | \sum \sigma_i \cdot \sum \sigma_j | 0 \rangle - 3A] \\
 &= -\frac{3}{A-1}
 \end{aligned}$$

Hence

$$M = -\frac{3}{A-1} \int d\mathbf{r}_1 \int d\mathbf{r}_2 \rho(\mathbf{r}_1, \mathbf{r}_2) t^{(s)}(\mathbf{r} - \mathbf{r}_1) t^{(s)}(\mathbf{r}' - \mathbf{r}_2)$$

or

$$\begin{aligned}
 M &= -\frac{3}{A-1} \left[ \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t^{(s)}(\mathbf{r} - \mathbf{r}_1) \right] \left[ \int d\mathbf{r}_2 \rho(\mathbf{r}_2) t^{(s)}(\mathbf{r} - \mathbf{r}_2) \right] \\
 &\quad - \frac{3}{A-1} \int d\mathbf{r}_1 \int d\mathbf{r}_2 C(\mathbf{r}_1, \mathbf{r}_2) t^{(s)}(\mathbf{r} - \mathbf{r}_1) t^{(s)}(\mathbf{r}' - \mathbf{r}_2)
 \end{aligned}$$

Define the density (but not correlation dependent) optical potential as follows:

$$\begin{aligned}
 \mathcal{V}_{\text{opt}}^{(1)} &= (A-1) \int \rho(\mathbf{r}_1) t^{(0)}(\mathbf{r} - \mathbf{r}_1) d\mathbf{r}_1 \delta(\mathbf{r} - \mathbf{r}') \\
 &\quad - 3(A-1) \langle \mathbf{r} | \frac{1}{\bar{\alpha}} | \mathbf{r}' \rangle \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t^{(s)}(\mathbf{r} - \mathbf{r}_1) \int d\mathbf{r}_2 \rho(\mathbf{r}_2) t^{(s)}(\mathbf{r}' - \mathbf{r}_2) \quad (4.45)
 \end{aligned}$$

The remainder of the optical potential  $V^{(1)} + V^{(2)} - \mathcal{V}^{(1)}$  will now involve only  $C(\mathbf{r}_1, \mathbf{r}_2)$  and not  $\rho$ .

The Schrödinger equation with the potential  $\mathcal{V}_{\text{opt}}^{(1)}$  is equivalent to a pair of coupled equations:

$$\begin{aligned}
 [E - K - (A-1) \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t^{(0)}(\mathbf{r} - \mathbf{r}_1)] \psi & \\
 = i\sqrt{3(A-1)} \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t^{(s)}(\mathbf{r} - \mathbf{r}_1) \phi & \\
 \bar{\alpha} \phi = [E - K - \bar{\epsilon} - (A-1) \int d\mathbf{r}_1 \rho(\mathbf{r}_1) t^{(0)}(\mathbf{r} - \mathbf{r}_1)] \phi & \\
 = i\sqrt{3(A-1)} \int d\mathbf{r}_1 \rho(\mathbf{r}_2) t^{(s)}(\mathbf{r} - \mathbf{r}_2) \psi & \quad (4.46)
 \end{aligned}$$



Equation (4.45) can be obtained by eliminating  $\phi$  from this pair of equations and identifying the potential in the resulting single-channel Schrödinger equation. The process being described consists of two scatterings. In the first the spin of the target nucleon is flipped by the  $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}_1$  term, generating the

amplitude  $\phi$ . The spin of the target and the amplitude  $\psi$  are restored through the action of the second  $\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}_2$ .

The magnitude of this spin effect will vary with circumstances, depending on the ratio between the first and second terms in (4.45), such as the strength of the spin-dependent amplitude compared to the spin-independent one. One can show that the ratio will decrease like  $1/\sqrt{E}$  but that energy dependence may be modified by the energy dependence of  $t^{(s)}$  and  $t^{(0)}$ . In any event, before the effects of correlations can be evaluated it is necessary to evaluate the spin- and isospin-dependent contributions which make their appearance in  $V_{\text{opt}}^{(2)}$ . The first order  $V_{\text{opt}}^{(1)}$  does not contain all the density dependence.

The optical potential  $\hat{V}_{\text{opt}}$  of (4.28) can be used to describe reaction processes such as inelastic scattering. In that case one simply includes the inelastic channels under study in (4.13). We consider the simple case of only one inelastic channel. One then obtains a pair of coupled equations with the potential matrix

$$\langle 0 | \hat{V}_{\text{opt}}^{(1)} | 0 \rangle, \quad \langle 0 | \hat{V}_{\text{opt}}^{(1)} | 1 \rangle, \quad \langle 1 | \hat{V}_{\text{opt}}^{(1)} | 0 \rangle, \quad \text{and} \quad \langle 1 | \hat{V}_{\text{opt}}^{(1)} | 1 \rangle,$$

We have just obtained  $\langle 0 | \hat{V}_{\text{opt}}^{(1)} | 0 \rangle$ , which in its local form, is given by (4.39), where  $\rho(\mathbf{r})$  is the density for the ground state,  $|0\rangle$ , of the target nucleus. The other diagonal component,  $\langle 1 | \hat{V}_{\text{opt}}^{(1)} | 1 \rangle$ , will have a similar structure with  $\rho(\mathbf{r})$  replaced by the density function for the excited state. The new elements are the coupling potentials, which will take the form

$$V_{01} \equiv \langle 0 | \hat{V}_{\text{opt}}^{(1)} | 1 \rangle = (A-1) \int \rho_{01}(\mathbf{r}_1) t(\mathbf{r} - \mathbf{r}_1) d\mathbf{r}_1 \quad (4.47)$$

where

$$\rho_{01}(\mathbf{r}_1) = \int \Psi_0^*(1, 2, \dots) \Psi_1(1, 2, \dots) d\mathbf{r}_2 \dots \quad (4.48)$$

Note that

$$\int \rho_{01}(\mathbf{r}_1) d\mathbf{r}_1 = 0 \quad (4.49)$$

as a consequence of orthogonality of the target wave functions.

The form factors  $\rho_{01}$ , as they are sometimes called, are discussed in some detail in Chapter V. We therefore will be content with a few remarks concerning the Fourier transform  $\tilde{\rho}_{01}(\mathbf{q})$ :

$$\tilde{\rho}_{01}(\mathbf{q}) = \int e^{i\mathbf{q} \cdot \mathbf{r}} \rho_{01}(\mathbf{r}) d\mathbf{r} \quad (4.50)$$

From (4.49) we have

$$\tilde{\rho}_{01}(0) = 0 \quad (4.51)$$

Moreover, if the transition from  $|0\rangle$  to  $|1\rangle$  involves a change in angular momentum, then  $\rho_{01}(\mathbf{r})$  will involve  $Y_{lm}(\theta, \phi)$ . Therefore, in the expansion of the plane wave  $\exp(i\mathbf{q}\cdot\mathbf{r})$  in (4.50) the first term that survives is proportional to  $j_l(qr)$ . As a consequence,  $\tilde{\rho}_{01}(q) \rightarrow q^l$  for small  $q$ . This is one of the effects of the angular momentum barrier. The Born approximation then indicates a sharp decrease in the transition amplitude as one approaches  $q=0$ . The effect of including distortion will not substantially modify this result, so that one expects the inelastic scattering cross section to have its maximum for  $qR \sim l$  and to decrease rapidly as  $q$  becomes smaller.

For most situations the distorted wave approximation (DWA) with interaction  $V_{01}$  of (4.47) suffices. The form of  $V_{01}$  suggests the possible use of folding to describe the coupling potential (see Chapter V). The extension to include correlation effects has been carried out [Feshbach (81)] and the reader is referred to that paper for more details.

## 5. THE SEMICLASSICAL APPROXIMATION<sup>†</sup>

This procedure is applicable when the wavelength of the projectile is small compared to the size of the system and when the projectile energy is much larger than the depth of the potential in which the projectile moves. We begin by considering the problem of the scattering of a projectile by a potential well and then develop the generalization to multiple scattering.

The propagation of short-wavelength radiation is a comparatively ancient subject which has received thorough study [van de Hulst (57)]. It occurs, for example, in the design of optical instruments, where the wavelength of the light is small compared to the size of the system. It is a method developed in that connection which we shall adapt to the present problem. Consider a wave propagating through a medium with an index of refraction  $n$ . The wavefront is defined to be an equiphase surface, while points on the wavefront trace out trajectories as the wave propagates. In the approximation to be used, these trajectories are calculated in the geometrical optics approximation (infinitely short wavelength) with the phase change along the ray given by the optical path length,  $\Delta\Phi$ :

$$\Delta\Phi = \int nk ds$$

where the integral is taken along the trajectory. By performing this calculation for each ray it is possible in principle to construct the equal-phase wavefronts and thereby follow the propagation of a wave through the medium.

In the present context, the ray is replaced by the classical mechanical

<sup>†</sup>Glauber (59); Feshbach (67).

trajectory of the particle. The index of refraction is given by

$$nk = \left[ \frac{2\mu}{\hbar^2} (E - V) \right]^{1/2} = (k^2 - U)^{1/2}$$

where

$$U \equiv \frac{2\mu}{\hbar^2} V \quad (5.1)$$

so that

$$\Delta\Phi = \int \sqrt{k^2 - U} ds \quad (5.2)$$

This approximation is thus a *semiclassical* one.

In the simplest form of the approximation it is assumed that the trajectories are straight that is, undeviated by the action of the potential and thus proceeding in the incident direction. This approximation requires that the momentum change which occurs because of the action of the potential be small compared to the initial momentum. Taking the force to be of the order of  $V/a$ , where  $a$  is an interaction length, and the time during which it acts as  $a/v$ , one obtains a momentum change of  $V/v$ . The resulting angular deflection is on the order of

$$\theta = \frac{V}{vp} = \frac{V}{2E} \quad (5.3)$$

The straight-ahead approximation is thus valid when

$$\theta \sim \frac{V}{2E} \ll 1 \quad (5.4)$$

To use the geometric approximation it is necessary that the classical trajectories be well defined. This requires that the classical momentum change  $V/v$  be larger than the quantum uncertainty in the momentum  $\hbar/R$ :

$$\frac{V}{v} \gg \frac{\hbar}{R}$$

or

$$\frac{VR}{\hbar v} = \frac{V}{2E} (kR) \gg 1 \quad (5.5)$$

We observe that this condition can be satisfied simultaneously with (5.4) only if

$$kR \gg 1 \quad (5.6)$$

**Problem.** Prove that the Born approximation is valid when  $(V/2E)(kR) \ll 1$ .

**Problem.** Show that in the relativistic regime (5.5) is replaced by  $VR/\hbar c \gg 1$ .

With the straight-ahead approximation it becomes a simple matter to calculate the phase change of a plane wave propagating in the  $z$  direction. Taking the initial phase ( $z \rightarrow -\infty$ ) to be  $kz$ , the phase at any  $z$  is

$$\Phi(z) = kz + \int_{-\infty}^z dz' (\sqrt{k^2 - U} - k)$$

and the corresponding “plane wave” by

$$\psi = \exp \left\{ i \left[ kz + \int_{-\infty}^z dz' (\sqrt{k^2 - U} - k) \right] \right\} \quad (5.7)$$

It should be noted that despite the straight-ahead approximation,  $\psi$  does describe a particle with both longitudinal and transverse momentum.

**Problem.** Let the solution,  $\psi$ , of the Schrödinger equation for potential scattering have the form

$$\psi = e^{iS}$$

Show that when  $\nabla^2 S \ll (\nabla S)^2$ ,

$$(\nabla S)^2 \simeq k^2 - U$$

Derive (5.7).

We are now in a position to evaluate the elastic scattering transition matrix from  $\psi$  to the plane wave with momentum  $\mathbf{k}_f$ :

$$\begin{aligned} \mathcal{T}_{e_i} &= \langle \phi_f^{(-)} | V \psi_i^{(+)} \rangle \\ &= \frac{\hbar^2}{2\mu} \int d\mathbf{r} e^{-i\mathbf{k}_f \cdot \mathbf{r}} U \exp \left\{ i \left[ \mathbf{k}_i \cdot \mathbf{r} + \int_{-\infty}^z dz' (\sqrt{k^2 - U} - k) \right] \right\} \\ &= \frac{\hbar^2}{2\mu} \int d\mathbf{r} e^{i(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{r}} U \exp \left\{ i \left[ \int_{-\infty}^z dz' (\sqrt{k^2 - U} - k) \right] \right\} \end{aligned} \quad (5.8)$$

To proceed further we choose the  $z$  axis to be along the direction  $(\mathbf{k}_i + \mathbf{k}_f)$ . This is a modification of the straight-ahead approximation. It is still assumed that the trajectories are straight lines but along the direction given by the average momentum  $(\mathbf{k}_i + \mathbf{k}_f)/2$ . Then  $(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{r}$  becomes  $(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{b}$  where  $\mathbf{b}$  is a vector

perpendicular to  $\mathbf{k}_i + \mathbf{k}_f$ . Evaluating this scalar product yields

$$(\mathbf{k}_i - \mathbf{k}_f) \cdot \mathbf{b} = 2 \sin \frac{\theta}{2} \cos \phi b k$$

Equation (5.8) becomes

$$\mathcal{F}_{\text{el}} = \frac{\hbar^2}{2\mu} \int dz \int d\mathbf{b} e^{2i \sin(\theta/2) \cos \phi b k} U \exp \left\{ i \left[ \int_{-\infty}^z dz' (\sqrt{k^2 - U} - k) \right] \right\}$$

The integration over the orientation of  $\mathbf{b}$  yields

$$\mathcal{F} = \frac{\pi \hbar^2}{\mu} \int dz \int b db J_0(2kb \sin \frac{1}{2} \theta) U \exp \left\{ i \left[ \int_{-\infty}^z dz' (\sqrt{k^2 - U} - k) \right] \right\} \quad (5.9)$$

A second approximation consistent with condition (5.4) permits the integration over  $z$  to be performed. Let

$$\Phi \equiv \sqrt{k^2 - U} - k$$

Then

$$U = -(2\Phi k + \Phi^2) \simeq -2\Phi k \quad (5.10)$$

with an error of  $U/k^2$  or  $V/E$ . Then

$$\mathcal{F}_{\text{el}} = -\frac{2\pi \hbar^2 k}{\mu} \int b db J_0(2kb \sin \frac{1}{2} \theta) \int_{-\infty}^{\infty} dz \Phi \exp \left( i \int_{-\infty}^z \Phi dz' \right)$$

The  $z$  integration can now be performed, yielding

$$\mathcal{F}_{\text{el}} = \frac{2\pi i \hbar^2 k}{\mu} \int_0^{\infty} b db J_0(2kb \sin \frac{1}{2} \theta) (e^{ix} - 1) \quad (5.11)$$

where

$$\chi(\mathbf{b}) = \int_{-\infty}^{\infty} \Phi dz = \int_{-\infty}^{\infty} dz (\sqrt{k^2 - U} - k) \quad (5.12)$$

This derivation avoids an expansion in the exponent and therefore leads to an error linear in  $V/E$  rather than  $\exp[i(V/E)]$ . With (5.11) we have thus reduced the calculation of  $\mathcal{F}_{\text{el}}$  to a quadrature with an error on the order of  $V/E$  and valid in the short-wavelength limit ( $kR \rightarrow \infty$ ) [see (5.4) and (5.5)]. The straight-line approximation used to obtain (5.9) requires, in addition, that the scattering angle  $\theta$  be small as stated by (5.4). A better limit is obtained from the requirement

that the value of  $k$  occurring in the expression for  $\chi(\mathbf{b})$  does not differ appreciably from the magnitude of  $\frac{1}{2}(\mathbf{k}_i + \mathbf{k}_f)$ . The condition that the error in  $\exp(i\chi)$  be small is

$$\theta < \frac{4E}{kRV} \quad (5.13)$$

**Problem.** In the expression (5.7) for  $\mathcal{F}_{el}$ , take the  $z$  direction to be along the incident direction. Replacing  $\exp[i(k - k \cos \theta)z]$  by unity, show that  $\mathcal{F}_{el}$  is given by (5.11) with, however,  $J_0(2kb \sin \frac{1}{2}\theta)$  replaced by  $J_0(kb \sin \theta)$ .

The expression for the elastic scattering amplitude

$$f_{el} = -\frac{1}{4\pi} \frac{2\mu}{\hbar^2} \mathcal{F}_{el} \quad (5.14)$$

is

$$f_{el} = \frac{k}{i} \int_0^\infty b db J_0(2kb \sin \frac{1}{2}\theta) (e^{ix} - 1) \quad (5.15)$$

Further exploitation of these results to be developed in this section depends on the result to be demonstrated now, that for real  $U$  (no absorption) unitarity is approximately satisfied (this is not the case for the Born approximation); that is, the total cross section,  $\sigma_T$ , given in that event by the total elastic cross section,

$$\sigma_{el} = 2\pi \int |f_{el}|^2 \sin \theta d\theta \quad (5.16)$$

is equal to the cross section calculated according to the expression

$$\sigma_T = \frac{4\pi}{k} \text{Im} f_{el}(0) \quad (5.17)$$

From (5.16) and (5.15) we have

$$\begin{aligned} \sigma_{el} &= 2\pi k^2 \int_0^\infty b db \int_0^\infty b' db' \int_0^\pi \sin \theta d\theta J_0(2kb \sin \frac{1}{2}\theta) J_0(2kb' \sin \frac{1}{2}\theta) \\ &\quad \times (e^{ix(b)} - 1)(e^{-ix(b')} - 1) \end{aligned} \quad (5.18)$$

We make use of the Fourier–Bessel integral [Morse and Feshbach (53, p. 766)]

$$\int_0^\infty J_0(\kappa b) J_0(\kappa b') \kappa d\kappa = \frac{\delta(b - b')}{b}$$

Letting  $\kappa = 2k \sin \frac{1}{2}\theta$ , this equation becomes

$$\int_0^{\pi - i\infty} J_0(2kb \sin \frac{1}{2}\theta) J_0(2kb' \sin \frac{1}{2}\theta) \sin \theta d\theta = \frac{\delta(b - b')}{k^2 b}$$

Assuming that  $(f_{e1})^2$  can be neglected well before one reaches  $\theta = \pi$ , the integration over  $\theta$  in (5.18) can be performed to yield

$$\sigma_{e1} = 2\pi \int_0^{\infty} b db |e^{ix(b)} - 1|^2 \quad (5.19)$$

while from (5.17)

$$\sigma_T = 4\pi \int_0^{\infty} b db (1 - \text{Re } e^{ix}) \quad (5.20)$$

As can be verified immediately,  $\sigma_T$  equals  $\sigma_{e1}$  when  $U$  and therefore  $\chi$  is real. The approximate satisfaction of the unitarity condition is thus demonstrated in the appropriate limit of no absorption.

When  $U$  is complex, absorption will occur. The absorption cross section  $\sigma_a$  can be obtained by subtracting  $\sigma_{e1}$ , (5.19), from  $\sigma_T$ , (5.20):

$$\sigma_a = 2\pi \int_0^{\infty} b db (1 - |e^{ix}|^2) = 2\pi \int_0^{\infty} b db (1 - e^{-2\text{Im } \chi}) \quad (5.21)$$

These formulas for the angle integrated cross sections [Eqs. (5.19), (5.20), and (5.21)] can be interpreted as being composed additively of contributions coming from a region between  $b$  and  $b + db$  with the area of  $2\pi b db$ . Each contribution can be calculated as if there is an  $S$  matrix, as a function of  $b$ , given by  $\exp(i\chi)$ . Indeed, these results can also be obtained from the phase shift series for  $\mathcal{T}_{e1}$  by taking a suitable high-energy limit. One then finds that

$$S \sim e^{2i\delta(b)}$$

where

$$\frac{1}{2}\chi(b) = \delta(b) \approx \delta_l \quad b = \frac{l + \frac{1}{2}}{k}$$

These results are very useful. Because of their simplicity, they permit a rapid evaluation of the elastic angular distributions as well as cross sections. Even in domains where they are not quantitatively valid, they yield qualitative results that are useful for orientation.

The results obtained with a square well [Feshbach (67); Bassichis, Feshbach

and Reading (71)] are instructive. From (5.15),

$$f(\theta) = \frac{iU_0}{2\kappa} \int_0^\infty b db J_0(2kb \sin \frac{1}{2}\theta) (e^{2i\kappa \sqrt{R^2 - b^2}} - 1) \quad (5.22)$$

where  $R$  is the radius of the well and

$$\kappa = \sqrt{k^2 - U_0} - k \xrightarrow{U_0 \ll k^2} -\frac{1}{2} \frac{U_0}{k}$$

The value of  $f$  at  $0^\circ$  is

$$f(0^\circ) = -\frac{iU_0 R^2}{2\kappa} \left\{ 1 + \left[ \frac{1}{2\kappa^2 R^2} - e^{2i\kappa R} \left( \frac{1}{i\kappa R} + \frac{1}{2\kappa^2 R^2} \right) \right] \right\} \quad (5.22')$$

In the limit  $\kappa R \rightarrow \infty$ , the optical theorem, (5.16), yields

$$\sigma_T \rightarrow 2\pi R^2 \quad (5.23)$$

As one can see from (5.20), this result can also be obtained when the absorption is so strong that  $e^{i\kappa x}$  can be neglected within the radius  $R$ . The absorption cross section is then

$$\sigma_a \rightarrow \pi R^2 \quad (5.24)$$

The angular distribution consists of two terms. The term, which is dominant near  $0^\circ$  generally and/or because of strong absorption, is given by the  $-1$  term in (5.22). Note that it is the  $U_0$  independent part of the integrand. We shall refer to it as the diffraction component,  $f_d$ . It contains that part of the scattered wave responsible for the formation of the shadow, as is immediately clear in the case of strong absorption. We find that

$$f_d(\theta) = -\frac{iU_0}{2\kappa} \int_0^R b db J_0(2kb \sin \frac{1}{2}\theta) = -\frac{iU_0}{2\kappa} R^2 \frac{J_1(2kR \sin(\theta/2))}{2kR \sin(\theta/2)} \quad (5.25a)$$

$$\rightarrow iR \frac{J_1(2kR \sin(\theta/2))}{2 \sin(\theta/2)} \quad (5.25b)$$

The angular distribution  $|f_d|^2$  obtained from (5.25b) has a strong maximum at  $\theta = 0^\circ$ :

$$f_d(0^\circ) = \frac{ikR^2}{2} \quad (5.25c)$$

The next maximum in  $|f_d|^2$  occurs at  $2kR \sin(\theta/2)$  equal to about 5.2, at which point the ratio to  $|f_d|^2$  at  $0^\circ$  is  $1/57.4$ , demonstrating the strength of the  $0^\circ$  maximum.

From (5.22') the value of the deviation from  $f_d(0^\circ)$  is given by

$$f_{e1}(0^\circ) - f_d(0^\circ) = \frac{iU_0 R^2}{2\kappa} \left[ \frac{e^{2i\kappa R}}{2i\kappa R} + \frac{1}{4\kappa^2 R^2} (e^{2i\kappa R} - 1) \right]$$

so that

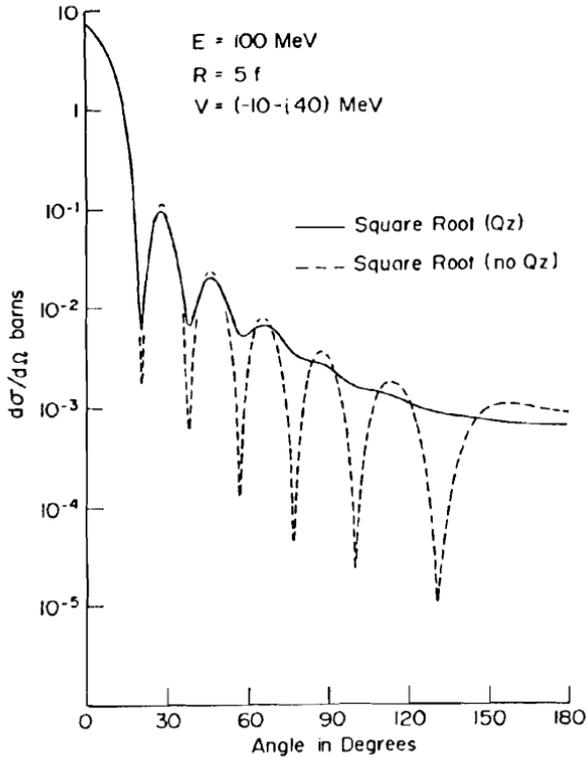
$$\frac{|f_{e1}(0^\circ) - f_d(0^\circ)|}{|f_d(0^\circ)|} \sim \frac{e^{-2\text{Im}\kappa R}}{|\kappa|R} \tag{5.26}$$

When  $\text{Im}|\kappa|R \gg 1$ , the small-angle scattering is dominated by  $f_d$ . Even when the absorption is small, the diffraction amplitude will make the major contribution when

$$|\kappa|R \approx \frac{|U_0|R}{2k} \gg 1 \tag{5.27}$$

Condition (5.27) is identical with the condition (5.5) that the Born approximation fail and that the classical trajectories are well defined. It is thus satisfied in the regime for which the approximation for  $\psi$ , (5.7), is valid.

At least two conditions must be met if the semiclassical method is to be applied to obtain cross sections for larger angles of scattering. The absorption must be sufficiently strong so that one can neglect scattering from the front surface of the scatterer. This could generate amplitudes that would interfere with the incident beam creating maxima and minima characteristics of "rainbow" scattering. Second, it no longer makes sense to use the straight-line approximation for, for example, scattering to the back angles. In the spirit of the semiclassical method, one should calculate the classical trajectories and then obtain a more accurate expansion for  $\psi$ . Equation (5.7) is no longer valid. The effect of neglecting the momentum transfer is shown in Fig. 5.1. The effect of expanding the square root  $\sqrt{k^2 - U_0}$  around  $k$  as used in (5.10) is shown in Fig. 5.2. Deviations from the exact result appear at  $\theta \sim |V|/E$ . With no absorption (Fig. 5.3) strong deviations appear even at small angles. These errors decrease in magnitude as the energy increases. For example, for scattering of 516-MeV nucleons (no absorption), the real and imaginary part of the amplitude is compared to the exact amplitude in Fig. 5.4. The semiclassical real part of the amplitude, longitudinal momentum neglected, fails after the first secondary maximum, while the imaginary part is incorrect even at  $0^\circ$  and increasingly beyond that point. In both cases there is agreement to within an order of magnitude with the exact results at back angles. Note that all of these examples use semilog abscissa. Figure 5.5 shows that the importance of absorption even



**FIG. 5.1.** Comparison between calculated cross sections to indicate the effect of including longitudinal momentum transfer and neglecting it, indicated by “no  $Q_z$ ” [From Bassichis, Feshbach, and Reading (71).]

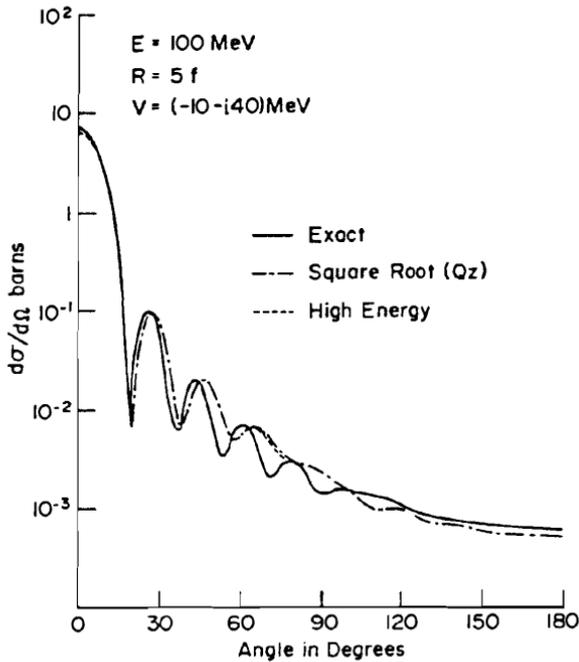
at higher energies, the exact result showing the large effect of interference at back angles.

The application of the semiclassical approximation to multiple scattering is referred to as the *Glauber approximation* [Glauber (59); McCauley and Brown (58)]. We begin with the elastic amplitude, (5.11), which using (5.14) becomes

$$\hat{f}_{el} = \frac{ik}{2\pi} \int e^{i\mathbf{q} \cdot \mathbf{b}} (1 - e^{i\chi(\mathbf{b}, k)}) d\mathbf{b} \quad (5.14)$$

The scattering is from  $A$  scatterers at positions  $\mathbf{r}_i (\equiv z_i, \mathbf{b}_i)$ , the scattering potential being given by

$$\mathcal{V} = \sum_i V(\mathbf{r} - \mathbf{r}_i) \quad (5.28)$$



**FIG. 5.2.** Differential cross section for the scattering of 100-MeV nucleons by a square-well potential  $5f$  in radius and potential  $V = (-10 - 40i)$  MeV. Comparison is made between the exact, the high-energy approximation (including the longitudinal momentum transfer indicated by “ $Qz$ ”), and the square-root form. [From Bassichis, Feshbach, and Reading (71).]

so that the phase function  $\chi$  is, according to (5.12),

$$\chi = \int_{-\infty}^{\infty} \left[ \sqrt{k^2 - \sum_i U(\mathbf{r} - \mathbf{r}_i)} - k \right] dz \quad (5.29)$$

where

$$U \equiv \frac{2\mu}{\hbar^2} V$$

Expanding the square root and keeping only the first term, one obtains

$$\chi(k, \mathbf{b}) = \sum \chi_i(k, \mathbf{b} - \mathbf{b}_i) \quad (5.30)$$

$$\chi_i = -\frac{1}{2k} \int_{-\infty}^{\infty} U_i dz = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} V(\mathbf{r} - \mathbf{r}_i) dz \quad (5.31)$$

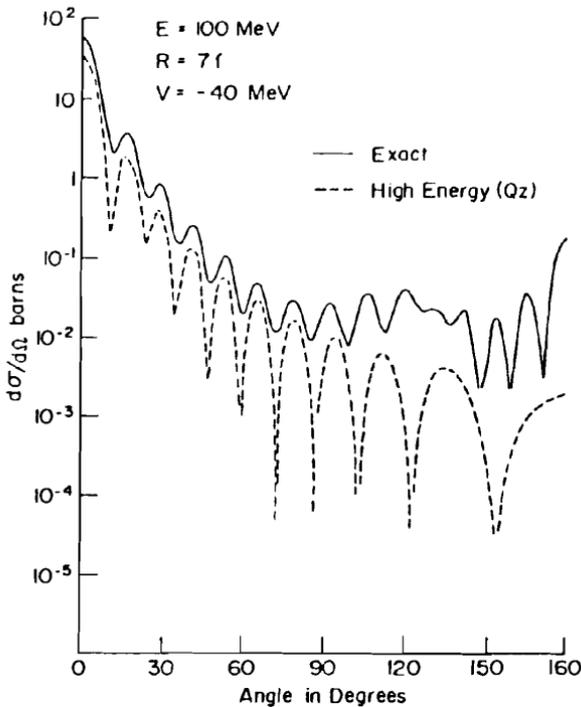


FIG. 5.3. Effect of absorption, where the radius of the square well is  $7f$ , the potential  $V = -40$  MeV, and the incident nucleon an energy of 100 MeV. [From Bassichis, Feshbach, and Reading (71).]

so that

$$\hat{f}_{el} = \frac{ik}{2\pi} \int d\mathbf{b} e^{i\mathbf{q}\cdot\mathbf{b}} \left( 1 - \prod_i e^{i\chi_i} \right) \quad (5.32)$$

Moreover,  $\chi_i$  may be related to the single scattering amplitude of the projectile by a fixed-target nucleon. In the same semiclassical approximation, it is

$$f_i(\mathbf{q}) = \frac{ik}{2\pi} \int e^{i\mathbf{q}\cdot\mathbf{b}} (1 - e^{i\chi_i}) d\mathbf{b} \quad (5.33)$$

Inverting this relation gives ( $\mathbf{q}_\perp =$  vector component along  $\mathbf{b}$ )

$$1 - e^{i\chi_i} = \frac{1}{2\pi ik} \int d\mathbf{q}_\perp e^{-i\mathbf{q}\cdot\mathbf{b}} f_i(\mathbf{q}) \quad (5.34)$$

where the integration is in the scattering plane containing the vectors  $\mathbf{k}_i$  and  $\mathbf{k}_f$ . The evaluation of this integral requires knowledge of  $f_i(\mathbf{q})$  for nonphysical

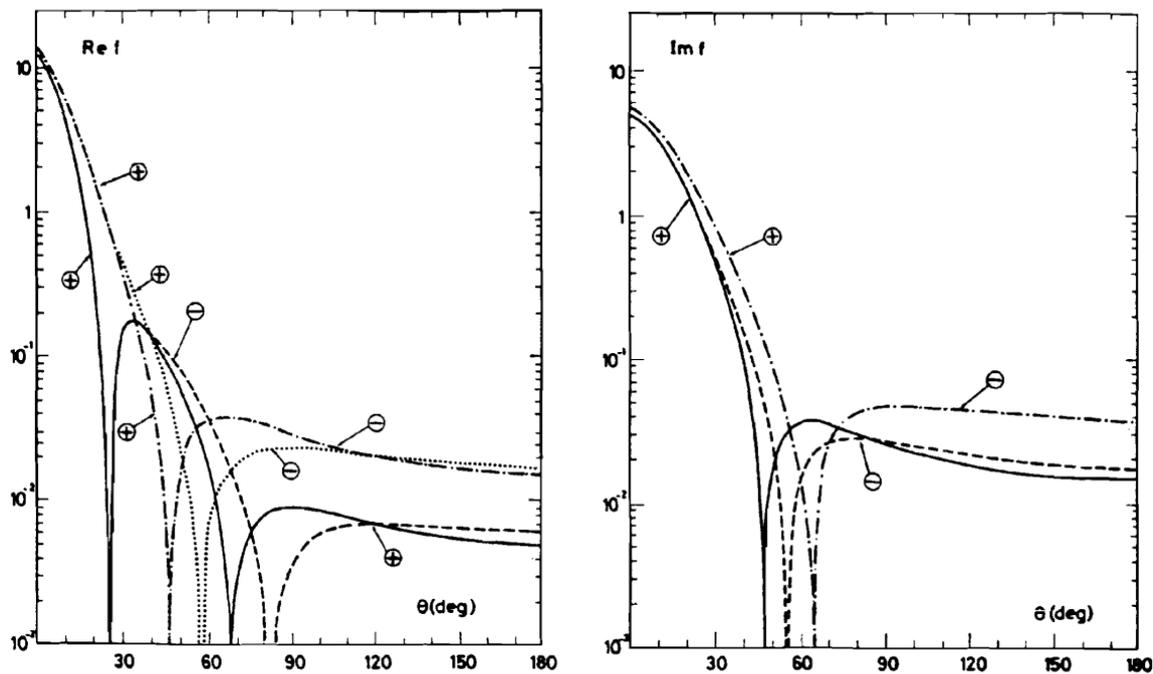


FIG. 5.4. Real and imaginary parts of the scattering amplitude induced by  $-V_0(e^{-r} - 1.125 e^{-2r})/r$ , where  $2m/\hbar^2 V_0 = 20$  and  $k = p/\hbar = 5$ . The solid line gives the exact result, the dashed curve the eikonal result, the dashed-dotted curve the second Born approximation, and the dotted curve the first Born approximations. [From Joachain (75).]

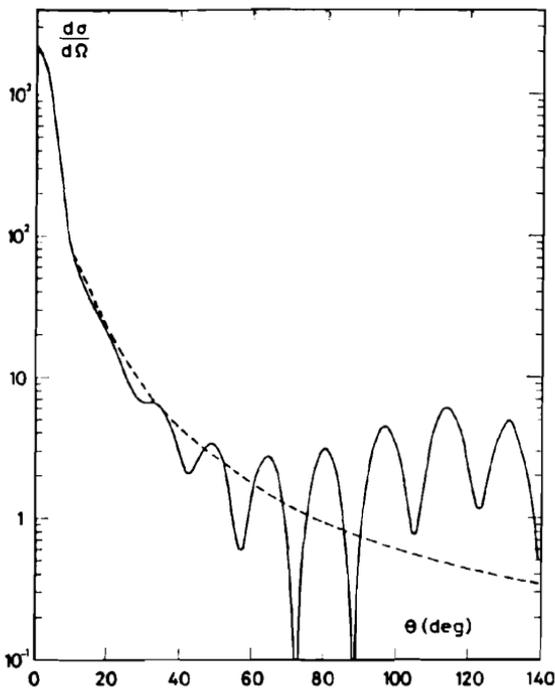


FIG. 5.5. Different cross section for a Yukawa potential,  $-V_0 e^{-r}/r$ , where  $(2m/\hbar^2)V_0 = 250$  and  $k = p/\hbar = 5$ . The solid curve shows the exact result, the dashed curve the eikonal result. [From Joachain (75).]

complex values  $|\cos \theta| > 1$  of the scattering angle  $\theta$ , since  $k$  is fixed. However, if  $f_i$  decreases rapidly enough with increasing  $q$ , this region is not expected to contribute appreciably to the integral.

In the Glauber approximation, one starts with (5.15), postulates additivity of the phases as given by (5.30), and computes  $e^{i\chi_i}$  from (5.34), avoiding any explicit mention of the scattering potential. Because of the additivity assumption, (5.30), the Glauber approximation assumes that the projectile is on the energy shell between collisions. It cannot, for example, include fully the effects of the collision of the projectile with two target nucleons (or more), since that will generally add terms in  $\chi$  that depend in a nonadditive fashion on the coordinates of both target nucleons. One can immediately see the presence of such a term by expanding the square root in (5.29) to second order. Then [Feshbach (69)]

$$\chi(k, \mathbf{b}) = \sum \chi_i + \sum_{i>j} w_{ij}(\mathbf{b} - \mathbf{b}_i, \mathbf{b} - \mathbf{b}_j, z_i - z_j) \quad (5.35)$$

where

$$w_{ij} = -\frac{1}{2k^3} \int_{-\infty}^{\infty} U(\mathbf{r} - \mathbf{r}_i) U(\mathbf{r} - \mathbf{r}_j) dz$$

As a consequence, the Glauber approximation in the form given so far cannot be used to evaluate the importance of correlations. The KMT formalism does take these two-body terms into account. They appear in the second-order term (4.41), which depends on the correlation function  $C(\mathbf{r}_1, \mathbf{r}_2)$ .

We return to (5.32). Because of (5.34) connecting  $\chi_i$  and  $f_i$  it is convenient to introduce the *profile function*

$$\Gamma_i \equiv 1 - e^{i\chi_i} \quad (5.36)$$

so that (5.32) is written

$$\hat{f}_{e1} = \frac{ik}{2\pi} \int d\mathbf{b} e^{i\mathbf{q}\cdot\mathbf{b}} \left[ 1 - \prod_i (1 - \Gamma_i) \right] \quad (5.37)$$

The scattering amplitude is obtained by taking the matrix element of  $\hat{f}_{e1}$  with respect to the ground state:

$$f_{e1} = \langle 0 | \hat{f}_{e1} | 0 \rangle = \frac{ik}{2\pi} \int d\mathbf{b} e^{i\mathbf{q}\cdot\mathbf{b}} \langle 0 | 1 - \prod_i (1 - \Gamma_i) | 0 \rangle \quad (5.38)$$

Note that the product,  $\prod_i (1 - \Gamma_i)$ , contains  $A$  factors, indicating that the target nucleons scatter the projectile nucleons only once. Expanding that product yields

$$1 - \prod_i (1 - \Gamma_i) = \sum_i \Gamma_i - \sum_{i \neq j} \Gamma_i \Gamma_j + \sum_{i \neq j \neq k} \Gamma_i \Gamma_j \Gamma_k + \dots \quad (5.39)$$

The first term yields the single scattering, the second the double scattering, and so on, ending with the  $A$  particle scattering ( $\Gamma_i \Gamma_j \Gamma_k \Gamma_l \dots \Gamma_A$ ) ( $i \neq j \neq k \dots$ ). According to (5.38), one must now take the expectation value of (5.39) with respect to the ground states:

$$\langle 0 | 1 - \prod_i (1 - \Gamma_i) | 0 \rangle = \langle 0 | \sum_i \Gamma_i | 0 \rangle - \sum_{i \neq j} \langle 0 | \Gamma_i \Gamma_j | 0 \rangle + \dots \quad (5.40)$$

When correlations, including those which are dynamic and those which are a consequence of the Pauli exclusion principle, are neglected, that is, using the independent-particle description for the target, (5.40) becomes

$$\begin{aligned} \langle 0 | 1 - \prod_i (1 - \Gamma_i) | 0 \rangle &= 1 - \prod_i \langle 0 | 1 - \Gamma_i | 0 \rangle = 1 - \prod_i \int \rho(\mathbf{r}_i) [1 - \Gamma(\mathbf{b} - \mathbf{b}_i)] d\mathbf{r}_i \\ &= 1 - \left[ \int \rho(\mathbf{r}_i) (1 - \Gamma(\mathbf{b} - \mathbf{b}_i)) d\mathbf{r}_i \right]^A \\ &= 1 - \left[ 1 - \int \rho(\mathbf{r}_i) \Gamma(\mathbf{b} - \mathbf{b}_i) d\mathbf{r}_i \right]^A \end{aligned}$$

Introducing the relation between the profile function  $\Gamma$  and the nucleon-projectile scattering amplitude  $f$  yields

$$f_{el} = \frac{ik}{2\pi} \int d\mathbf{b} e^{i\mathbf{q} \cdot \mathbf{b}} \left\{ 1 - \left[ 1 - \frac{1}{2\pi ik} \int \tilde{\rho}(\mathbf{q}', 0) e^{-i\mathbf{q}' \cdot \mathbf{b}} f(\mathbf{q}') d\mathbf{q}' \right]^A \right\} \quad (5.41)$$

where  $\tilde{\rho}(\mathbf{q}, 0)$  is the Fourier transform of the density with the momentum transfer along the longitudinal direction,  $(\mathbf{k}_i + \mathbf{k}_f)/2$ , put equal to zero.

If the binomial in (5.41) is expanded, the first surviving term in (5.41) is proportional to  $A$ , the second to  $A(A-1)/2$ , and so on; the term proportional to  $A$  is the contribution to the amplitude from single scattering, and the term proportional to  $A(A-1)/2$  is the contribution of the scattering from two target nucleons. When the projectile-nucleon amplitude is sharply peaked in the forward direction in the laboratory system, as is the case for high-energy projectiles, one can readily see that the double scattering term has a wider angular dispersion than the single scattering term. Thus in this picture the first diffraction peak comes from the double scattering term, while the first diffraction minimum is a consequence of destructive interference between the single and double scattering terms. Equation (5.41) is a remarkably simple result that can readily be evaluated to obtain the elastic scattering amplitude. Its validity is restricted to forward scattering, which is most probable for high-energy projectiles. In view of the additivity assumption [Eq. (5.30)] it neglects correlations arising from the Pauli principle as well as those coming from the nature of the interaction. Equation (5.41) assumes that a nucleon in the target nucleus scatters the projectile only once. For these reasons it is most appropriate for a low-density target system.

When  $A$  is large, [brackets] <sup>$A$</sup>  in (5.41) can be approximated as follows:

$$\left[ 1 - \frac{1}{2\pi ik} \int \tilde{\rho}(\mathbf{q}, 0) e^{-i\mathbf{q} \cdot \mathbf{b}} f(\mathbf{q}) d\mathbf{q} \right]^A \rightarrow e^{i\chi_A(\mathbf{b})} \quad (5.42)$$

where

$$\chi_A = \frac{A}{2\pi k} \int \tilde{\rho}(\mathbf{q}, 0) e^{-i\mathbf{q} \cdot \mathbf{b}} f(\mathbf{q}) d\mathbf{q} \quad (5.43)$$

For large nuclei  $\tilde{\rho}(\mathbf{q}, 0)$  is sharply peaked around  $\tilde{\mathbf{q}} = 0$ , so that

$$\begin{aligned} \chi_A &\simeq \frac{Af(0)}{2\pi k} \int \tilde{\rho}(\mathbf{q}, 0) e^{-i\mathbf{q} \cdot \mathbf{b}} d\mathbf{q} \\ &= \frac{2\pi Af(0)}{k} \int dz \rho(\mathbf{r}) \end{aligned} \quad (5.44)$$

where  $z$  is in the direction perpendicular to  $\mathbf{b}$ , that is, in the direction of  $\frac{1}{2}(\mathbf{k}_i + \mathbf{k}_f)$ .

The function  $T(\mathbf{b})$

$$T(\mathbf{b}) = \int dz \rho(\mathbf{r}) \quad (5.45)$$

is referred to as the *thickness function*, since it gives the thickness of the target nucleus as a function of the impact parameter  $\mathbf{b}$ .

With approximation equation (5.42), (5.41) can be written

$$f_{el} = \frac{ik}{2\pi} \int d\mathbf{b} e^{i\mathbf{q}\cdot\mathbf{b}} (1 - e^{i\chi_A}) \quad (5.46)$$

which has the form to be expected from an optical potential model, (5.14). One may then calculate the total absorption cross section,  $\sigma_a$ , according to (5.21):

$$\sigma_a = \int d\mathbf{b} (1 - e^{-2\text{Im}\chi_A})$$

But  $\text{Im}\chi_A$ , in the large-nucleus approximation, is, from (5.44),

$$2\text{Im}\chi_A = \frac{4\pi A T(\mathbf{b})}{k} \text{Im}f(0) = A\sigma T(\mathbf{b}) \quad (5.47)$$

where we have used  $4\pi/k \text{Im}f(0) = \sigma$ , the cross section for the projectile-nucleon cross section. Therefore,

$$\sigma_a = \int d\mathbf{b} (1 - e^{-A\sigma T(\mathbf{b})}) \quad (5.48)$$

the classical result.

Once  $\chi_A$  is known, (5.43), one can ask for the equivalent optical model potential, that is, the potential that will give rise to the known  $\chi_A$  through the relation (5.31):

$$\chi_A(\mathbf{b}) = -\frac{1}{2k} \int_{-\infty}^{\infty} U_{\text{opt}}(\mathbf{r}) dz = -\frac{1}{\hbar v} \int_{-\infty}^{\infty} V_{\text{opt}}(\mathbf{r}) dz \quad (5.49)$$

Using the approximate expression for  $\chi_A$ , (5.44), one obtains

$$U_{\text{opt}} = -4\pi A f(0) \rho(\mathbf{r}) \quad (5.50)$$

This should be compared with the result obtained using the KMT method of Section 4, which yields, according to (4.40) and using the large-nucleus

approximation,

$$U_{\text{opt}} = -4\pi(A-1)f(0)\rho(\mathbf{r})$$

**Problem.** One can consider (5.49) as an integral equation for  $U_{\text{opt}}$ . With the assumption  $U_{\text{opt}} = U_{\text{opt}}(\mathbf{r})$ , this equation may be solved. Toward this end, use  $r$  as the integration variable in (5.49), so that it becomes

$$\chi_A(b) = -\frac{1}{k} \int_b^\infty \frac{U(r)r dr}{\sqrt{r^2 - b^2}}$$

This is the Abel integral equation. Show that the solution is

$$U(r) = \frac{2k}{\pi} \int_r^\infty \frac{db}{\sqrt{b^2 - r^2}} \frac{d\chi_A}{db} \quad (5.51)$$

Note the result

$$\int_R^r \frac{b db}{\sqrt{(r^2 - b^2)(b^2 - R^2)}} = \frac{\pi}{2}$$

Discuss (5.51) using a reasonable description of  $\chi_A$ .

In this section and the preceding one, we have developed two different formalisms, the KMT and the Glauber approximations, for the multiple scattering of high-energy projectiles by a target nucleus. A comparison between the two procedures is possible for the formulas for the KMT  $V_{\text{opt}}^{(1)}$ , (4.40), and the Glauber  $f_{\text{el}}$ , (5.41). Diagrammatically, both of these correspond to a component of the multiple scattering in which the target nucleus and the projectile are never excited, as illustrated in Fig. 5.6 for the scattering amplitude where the vertical lines indicate the presence of an interaction. The second KMT term,  $V_{\text{opt}}^{(2)}$ , (4.41), correspond to the contribution in which the target nucleus is excited and then deexcited, as shown in Fig. 5.7. The Schrödinger equation involving  $V_{\text{opt}}^{(1)} + V_{\text{opt}}^{(2)}$  iterates the two elementary diagrams, the one shown in Fig. 5.7 and the diagram in Fig. 5.8, which is the basis for Fig. 5.6.

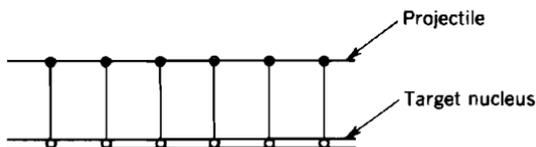


FIG. 5.6

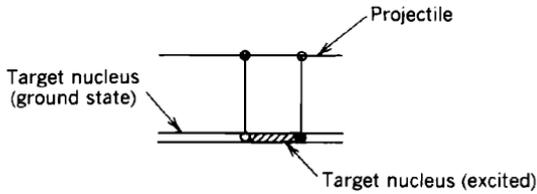


FIG. 5.7

Unfortunately, for the hope of using strongly interacting hadronic probes to study correlations, studies of the high-energy proton reactions have failed so far to reveal any easily identifiable and substantial effects (see Chapter IX) of the correlation terms in elastic scattering, so that the KMT equation (4.40) and the Glauber equation (5.41) do suffice for most purposes. For this contribution we can think of (5.41) as providing a solution to the Schrödinger equation for the optical potential. It is a convenient solution particularly for small systems, certainly more readily evaluated than a phase-shift analysis when the energy is large (unless of course the WKB method is used). It is, however, approximate and is not accurate at the larger angles or at the diffraction minima.

The effect of correlations can be introduced into the Glauber approximation by adding two-body terms to the expression for the phase-shift function  $\chi$  as in (5.35). Some of the consequences of that *ansatz* have been developed [Feshbach (69)]. One of these is qualitatively important. In obtaining (5.41), correlations were neglected. In particular,  $\langle 0 | \sum_{i \neq j} \Gamma_i \Gamma_j | 0 \rangle$  was placed equal to  $\sum_{i \neq j} \langle 0 | \Gamma_i | 0 \rangle \langle 0 | \Gamma_j | 0 \rangle$ . The difference,  $\sum_{i \neq j} [\langle 0 | \Gamma_i \Gamma_j | 0 \rangle - \langle 0 | \Gamma_i | 0 \rangle \langle 0 | \Gamma_j | 0 \rangle]$ , involves correlations. However, when two-body terms are included in  $\chi$ , one obtains, instead,  $\langle 0 | \Gamma_i \Gamma_j + i w_{ij} | 0 \rangle$ , so that in discussing correlation effects one must take into account both the on-the-energy-shell effects given by  $\Gamma_i \Gamma_j$  and the effect of dynamical correlations as described by  $w_{ij}$ . If one takes for  $w_{ij}$  the form given in (5.35) obtained by expanding the square root  $(k^2 - U)^{1/2}$ , one finds that the  $w_{ij}$  term gives the effect of the overlap of the potentials acting between the incident projectile and two of the target nucleons. In this model the effect of overlap does tend to zero with increasing energy, so that at sufficiently large energies the  $w_{ij}$  term should be relatively unimportant. The important point to be borne in mind is that it is not possible to distinguish between the correlation effects present in the ground-state wave functions and the effect of overlapping potentials. The two mechanisms give rise to indistinguishable matrix elements (except for their energy dependence).

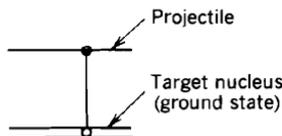


FIG. 5.8

## 6. CENTER-OF-MASS AND PAULI-PRINCIPLE CORRELATIONS, FERMI MOTION

### A. Center-of-Mass Correlations

The effect of center-of-mass correlations is particularly important for light nuclei. It is a consequence of the conservation of momentum, which requires that the momentum of the center of mass be unchanged by the interaction between the projectile and the nucleus. This requirement is formally satisfied by target nuclear wave functions that depend only on intrinsic coordinates, that is, on

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R} \quad (6.1)$$

where  $\mathbf{r}_i$  is the coordinate of  $i$ th target-nucleus nucleon and  $\mathbf{R}$  is the target-nucleus center of mass. Similarly, the wave function for the projectile–nucleus system depends only on the coordinate of the projectile relative to the center of mass of the nucleus. However, it is very often the case that the model wave functions  $\Psi^{(M)}$  available for the calculation of  $\rho(\mathbf{r})$  and  $C(\mathbf{r}, \mathbf{r}')$  (e.g., the interacting shell model wave functions) have not had their center-of-mass motion removed; that is, they are written as functions of the  $3A$  coordinates  $\mathbf{r}_i$  rather than of the  $3(A-1)$  coordinates  $\mathbf{r}'_i$  of (6.1). To the extent that the model wave functions are good, so that an approximate decoupling of the center-of-mass motion and the internal motion occurs, one can relate the model density and correlation functions with the exact  $\rho$  and  $C$ .

We begin with  $\rho$  and recall that

$$\tilde{\rho}^{(M)}(\mathbf{q}) = \langle \Psi_0^{(M)} | e^{i\mathbf{q}\cdot\mathbf{r}_1} \Psi_0^{(M)} \rangle \quad (6.2)$$

where the superscript ( $M$ ) indicates model quantities. Introducing (6.1), one finds that

$$\tilde{\rho}^{(M)}(\mathbf{q}) = \langle \Psi_0^{(M)} | e^{i\mathbf{q}\cdot\mathbf{r}'_1} e^{i\mathbf{q}\cdot\mathbf{R}} | \psi_0^{(M)} \rangle$$

If  $\Psi_0^{(M)}$  were exact, it would be a product wave function:

$$\Psi_0^{(M)} = \psi_0(\mathbf{r}'_1, \mathbf{r}'_2, \dots) \varphi_{\text{cm}}(\mathbf{R}) \quad (6.3)$$

Then  $\tilde{\rho}^{(M)}(\mathbf{q})$  would factor as

$$\tilde{\rho}^{(M)}(\mathbf{q}) = \tilde{\rho}(\mathbf{q}) \tilde{\rho}_{\text{cm}}(\mathbf{q})$$

where

$$\tilde{\rho}(\mathbf{q}) = \langle \psi_0(\mathbf{r}'_1 \dots) | e^{i\mathbf{q}\cdot\mathbf{r}'_1} \psi_0(\mathbf{r}'_1 \dots) \rangle$$

and

$$\tilde{\rho}_{\text{cm}}(\mathbf{q}) = \langle \varphi_{\text{cm}} | e^{i\mathbf{q}\cdot\mathbf{R}} \varphi_{\text{cm}} \rangle \simeq \langle \Psi_0^{(M)} | e^{i\mathbf{q}\cdot\mathbf{R}} \Psi_0^{(M)} \rangle. \quad (6.4)$$

Therefore, the desired  $\tilde{\rho}(\mathbf{q})$  is

$$\tilde{\rho}(\mathbf{q}) = \frac{\tilde{\rho}^{(M)}(\mathbf{q})}{\tilde{\rho}_{\text{cm}}(\mathbf{q})} \quad (6.5)$$

In case the wave functions used for  $\Psi_0^{(M)}$  are constructed from single-particle harmonic oscillator wave functions, (6.3) and (6.5) are exact. [For details, see Feshbach, Gal, and Hufner (71).]

For the correlation function  $\tilde{C}(\mathbf{q}, \mathbf{q}')$  we have

$$\begin{aligned} \tilde{C}^{(M)}(\mathbf{q}, \mathbf{q}') &= \frac{1}{A(A-1)} \sum_{i \neq j} \langle \Psi_0^{(M)} | e^{i\mathbf{q} \cdot \mathbf{r}_i} e^{i\mathbf{q}' \cdot \mathbf{r}_j} | \Psi_0^{(M)} \rangle \\ &= \langle \Psi_0^{(M)} | e^{i\mathbf{q} \cdot \mathbf{r}_1 + i\mathbf{q}' \cdot \mathbf{r}_2} | \Psi_0^{(M)} \rangle - \tilde{\rho}^{(M)}(\mathbf{q}) \tilde{\rho}^{(M)}(\mathbf{q}') \end{aligned} \quad (6.6)$$

Introducing the factorization, (6.3), one finds that

$$\tilde{C}^{(M)}(\mathbf{q}, \mathbf{q}') = \tilde{C}(\mathbf{q}, \mathbf{q}') \tilde{\rho}_{\text{cm}}(\mathbf{q} + \mathbf{q}') + \tilde{\rho}(\mathbf{q}) \tilde{\rho}(\mathbf{q}') [\tilde{\rho}_{\text{cm}}(\mathbf{q} + \mathbf{q}') - \tilde{\rho}_{\text{cm}}(\mathbf{q}) \tilde{\rho}_{\text{cm}}(\mathbf{q}')] \quad (6.7)$$

From this equation  $\tilde{C}$  is readily obtained, using (6.5), in terms of model quantities.

In light nuclei, the center-of-mass effect can be substantial, as illustrated by Fig. 6.1. The importance of center-of-mass correlations for elastic scattering decreasing rapidly with increasing  $A$  and is not visible for nucleon- $^{16}\text{O}$  scattering [Feshbach, Gal, and Hufner (71)].

## B. Pauli Correlations

The Pauli exclusion principle requires that the wave function for the target nucleus be antisymmetric. As a consequence, even in the absence of a residual interaction, correlations are implied. As a first example we use a Slater determinant for a  $p$ -shell nucleus. The  $1s$  and  $1p$  orbitals are taken to be the harmonic oscillatory wave functions

$$\varphi_{1s}(\mathbf{r}) \sim e^{-(v/2)r^2} \quad \varphi_{1p}(\mathbf{r}) \sim r Y_{1m}(\vartheta, \varphi) e^{-(v/2)r^2}$$

One then finds that [Feshbach, Gal, and Hufner (71); Lambert and Feshbach (73)]

$$\tilde{\rho}^{(M)}(\mathbf{q}) = \left( 1 - \frac{A-4}{6Av} q^2 \right) e^{-q^2/4v}$$

and

$$\tilde{C}^{(M)}(\mathbf{q}, \mathbf{q}') = \left[ a_0 \frac{\mathbf{q} \cdot \mathbf{q}'}{v} + a_1 \left( \frac{qq'}{v} \right)^2 - a_2 \left( \frac{\mathbf{q} \cdot \mathbf{q}'}{v} \right)^2 \right] e^{-(q^2 + q'^2)/4v}$$

when the values of the coefficients  $a_n$  are as given in Table 6.1.

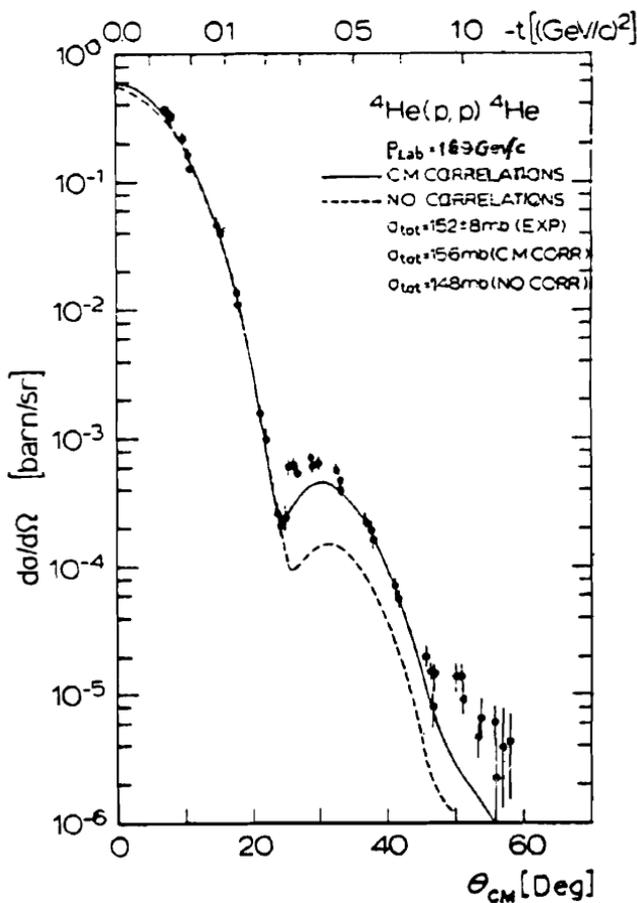


FIG. 6.1. Effect of CM correlations in  $p$ - ${}^4\text{He}$  elastic scattering. The solid line includes this effect, the dashed does not. Disregard experimental points, which have been changed substantially in later experiments. [From Feshbach, Gal, and Hufner (71).]

As indicated, the Pauli correlations for  ${}^4\text{He}$  vanishes for this model wave function, as the exclusion principle has no effect. In view of the small value of the coefficients, one can expect that the Pauli correlations will have little effect on the elastic scattering. This expectation is borne but by calculations for the smaller scattering angles. However, some effects do appear beyond the first secondary maximum.

Another model appropriate for heavy target nuclei, is the Fermi-gas model. The two-body correlation for that case has been derived in Chapter II [Eq. (II.5.14)] of deShalit and Feshbach (74). This leads to the following result, after averaging over spin and isospin:

$$C(\mathbf{r}_1, \mathbf{r}_2) = -\rho(\mathbf{r}_1)\rho(\mathbf{r}_2) \frac{A}{4(A-1)} \left[ \frac{3j_1(k_F|\mathbf{r}_1 - \mathbf{r}_2|)}{k_F|\mathbf{r}_1 - \mathbf{r}_2|} \right]^2 \quad (6.8)$$

TABLE 6.1

	${}^4\text{He}$	${}^{12}\text{C}$	${}^{16}\text{O}$
$a_0$	0	$\frac{2}{99}$	$\frac{1}{60}$
$a_1$	0	$\frac{1}{3564}$	$\frac{1}{120}$
$a_2$	0	$\frac{1}{960}$	$\frac{1}{240}$

where  $j_1$  is the spherical Bessel function of first order. As commented on in Chapter II of deShalit and Feshbach (74), the net correlation is repulsive (Pauli repulsion); that is, it has the effect of increasing the average distance between target nucleons beyond what would be calculated from a simple product wave function. Approximating the term in brackets in (6.8) by the form (B.13) (see Appendix B at the end of this chapter) yields an effective<sup>†</sup>  $r_c$  of  $5k_F/\sqrt{6}$ , from which a length defined by (B.14) can be obtained, noting that  $\beta \ll r_c$ . Finally, the second-order potential can be obtained from (B.12). The effect is not small, so that Pauli correlations are of importance for the heavier nuclei. Calculations show that effects of this order of magnitude are visible at the larger angles.

### C. Fermi Motion

The nucleons in the nucleus are moving. So far the discussion has assumed them to be stationary because during the passage of the projectile through the nucleus with a velocity close to  $c$ , the velocity of light, the nucleons in the target nucleus hardly move. The motion of the nucleus can be taken into account in the first-order potential  $V_{\text{opt}}^{(1)}$ , (4.32) or (4.37). For this purpose one needs the density matrix  $K(\mathbf{r}, \mathbf{r}')$  as well as the projectile-target nucleon transition  $t$  matrix. The first is model dependent. The second involves the  $t$  matrix off the energy shell, therefore requiring a complete description of the projectile-nucleon interaction for its determination. The density matrix  $K(\mathbf{r}, \mathbf{r}')$  in the independent-particle approximation is given by

$$K(\mathbf{r}, \mathbf{r}') = \sum_{\lambda=1}^A \psi_{\lambda}(\mathbf{r})\psi_{\lambda}^*(\mathbf{r}') \quad (6.9)$$

where  $\psi_{\lambda}$  are the single-particle orbitals. In the Fermi-gas model

$$K(\mathbf{r}, \mathbf{r}') = \rho_0 \frac{3j_1(k_F|\mathbf{r} - \mathbf{r}'|)}{k_F|\mathbf{r} - \mathbf{r}'|} \quad (\text{Fermi-gas model}) \quad (6.10)$$

where  $\rho_0$  is the density ( $A/\Omega$ ). The major effect of the Fermi motion is to introduce a nonlocality with a range of the order of  $1/k_F$  into the first-order

<sup>†</sup>The quantities  $\beta$  and  $r_c$  are defined in Appendix B of this chapter.

optical potential. However, as can be seen from (4.37) and the ensuing discussion, the behavior of  $K(\mathbf{r}, \mathbf{r}')$ ,  $\mathbf{r} \neq \mathbf{r}'$  will be important only if  $t(\mathbf{r}, \mathbf{r}')$ , the elementary amplitude, is also nonlocal. If  $t(\mathbf{r}, \mathbf{r}')$  is local, that is, proportional to  $\delta(\mathbf{r} - \mathbf{r}')$  as in (4.38), only  $K(\mathbf{r}, \mathbf{r}) \equiv \rho_0$  survives and there is then no impact of the Fermi motion on the first-order optical potential. We estimate that the influence of the Fermi motion is determined by the parameter  $(k_F a)^2/30$ , where  $a$  is the range of the nonlocality in  $t(\mathbf{r}, \mathbf{r}')$ . In the case of nucleon–nucleon scattering,  $a \sim 0.7$  fm, so that the magnitude of the Fermi-motion term is on the order of several percent. It is clear that Fermi motion becomes more significant in the presence of long-range nonlocal elementary projectile–nucleon amplitudes.

## 7. SOME KINEMATICS

(a) One rather obvious requirement of importance for high-energy projectiles is the appropriate use of relativity and at the very least of relativistic kinematics. The preferred method up to recently [Goldberger and Watson (64); Kerman, McManus, and Thaler (59)] has been to insert the optical potential of Section II.4 in deShalit and Feshbach (74) into the Klein–Gordon equation. There is an ambiguity even in this simplistic procedure since one must postulate the transformation properties of the optical potential. One obtains different results, as we shall see, according to whether one presumes that the potential is the fourth component of a 4-vector or a scalar. In a recent development, a covariant description of the  $t$  matrix is used and the optical model employs the Dirac equation when the projectile is a nucleon. Use of the Dirac optical model is presented in Chapter V. In the following discussion we follow the derivation of Goldberger and Watson (64) and Kerman, McManus, and Thaler (59).

Goldberger and Watson begin with the expression of the energy in the center-of-mass frame assuming that the optical potential  $V$  is the fourth component of a four-vector. Let the energy of the system, *excluding the rest mass of the target nucleus but including the rest mass of the projectile* be  $\mathcal{E}$ . Then

$$(\mathcal{E} - V) + A m c^2 = \sqrt{m_p^2 c^4 + p^2 c^2} + A m c^2 + \frac{c^2 p^2}{2 A m c^2} \quad (7.1)$$

where the energy of the target nucleus is expressed nonrelativistically.  $A$  is the mass number of the target and  $m_p$  is the mass of the projectile. Solving this equation for  $p^2$  to first order in  $V$  (i.e., taking  $\mathcal{E} \gg V$ ) yields

$$\begin{aligned} c^2 p^2 &= (\mathcal{E}^2 - m_p^2 c^4) \frac{1}{1 + (\mathcal{E} - V)/A m c^2} - 2 \mathcal{E} V \frac{1}{1 + (\mathcal{E} - V)/A m c^2} \\ &\simeq (\mathcal{E}^2 - m_p^2 c^4) \frac{1}{1 + \mathcal{E}/A m c^2} - 2 \mathcal{E} V \frac{1}{1 + \mathcal{E}/A m c^2} \end{aligned}$$

Replacing  $\mathbf{p}$  by  $\hbar/i\nabla$  yields the following Schrödinger-type equation:

$$\left[ \nabla^2 + \frac{1}{\hbar^2 c^2} \left( \frac{\mathcal{E}^2 - m_p^2 c^4}{1 + \mathcal{E}/Amc^2} - 2\mathcal{E}V \frac{1}{1 + \mathcal{E}/Amc^2} \right) \right] \psi = 0 \quad (7.2)$$

The (wave number)<sup>2</sup>,  $k^2$ , which is given in the Schrödinger case by the nonrelativistic  $2Am m_p E/\hbar^2(m_p + mA)$  (where  $E = \mathcal{E} - m_p c^2$ ) is replaced by  $(\mathcal{E}^2 - m_p^2 c^4)/\hbar^2 c^2 (1 + \mathcal{E}/Amc^2)$ . In the nonrelativistic limit ( $E \ll mc^2$ ) the last expression reduces to the nonrelativistic value. We note that the effective potential is energy dependent. This is a result of the assumption that  $V$  is the fourth component of a 4-vector. If  $V$  is a scalar, that energy dependence is not present.

**Problem.** Suppose that  $V$  is a scalar. Show that the only change in (7.2) is the replacement of  $\mathcal{E}V$  by  $mc^2 V$ . Derive the Schrödinger equation when  $V$  has two components,  $V_0$  and  $V_s$ , where  $V_0$  is the fourth component of a 4-vector and  $V_s$  is a scalar.

(b) The transition matrix elements,  $\tilde{t}$ , for elastic projectile-nucleon scattering are usually given with respect to the projectile-nucleon center-of-mass frame. Multiple scattering theory requires their value in the projectile-nucleus frame. The transformation between the two frames is governed by the result that

$$\sqrt{E_1' E_2'} \langle \mathbf{p}_1', \mathbf{p}_2' | \hat{\mathcal{T}} | \mathbf{p}_1, \mathbf{p}_2 \rangle \sqrt{E_1 E_2} \quad \text{is an invariant} \quad (7.3)$$

In this expression  $(\mathbf{p}_1, E_1)$ ,  $(\mathbf{p}_2, E_2)$  are the momentum and energy of each particle (projectile, nucleon) before collision, and  $(\mathbf{p}_1', E_1')$ ,  $(\mathbf{p}_2', E_2')$  are their values after collision. The wave functions in the matrix element of  $\hat{\mathcal{T}}$  are assumed to be in the form  $\exp(i\mathbf{p}\cdot\mathbf{r})/\hbar$  asymptotically with unit amplitude. However, since the invariant volume in momentum space is  $d\mathbf{p}/E$ , the invariant normalization is given by

$$\langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi\hbar)^3 \delta(\mathbf{p}' - \mathbf{p})/E$$

This condition leads immediately to the result (7.3).

Let  $t_{pn}$  be the value of the matrix element of  $\hat{\mathcal{T}}$  in the projectile-nucleon center-of-mass frame, while  $t_{PN}$  is its value in the projectile-nucleus frame. From (7.3) these are related by

$$E_1 E_2 t_{PN} = \varepsilon_1 \varepsilon_2 t_{pn} \quad (7.4)$$

where  $E_1$  and  $E_2$  are the energies of the projectile and target nucleon, including their rest masses in the projectile-nucleus frame, and  $\varepsilon_1$  and  $\varepsilon_2$  are the corresponding energies in the projectile-nucleon frame. Equation (7.4) is

approximate since  $E_1 \neq E'_1, E_2 \neq E'_2$ . The equation is valid for small momentum transfers; the error is on the order of  $q^2(p^2 - q^2)/m^2 E^2$ . In addition,  $t_{pn}$  is related to the scattering amplitude  $f$  in that frame by

$$t_{pn} = -\frac{4\pi\hbar c^2}{E_0} f_{pn} \tag{7.5}$$

where<sup>†</sup>

$$E_0 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) \tag{7.6}$$

Combining (7.5) and (7.4) yields

$$t_{PN} = -4\pi\hbar^2 c^2 \frac{\varepsilon_1 \varepsilon_2}{E_1 E_2} \frac{1}{E_0} f_{pn} \tag{7.7}$$

We now relate all the energies  $\varepsilon_1, \varepsilon_2$ , and so on, to the energy  $E_L$  and momentum  $p_L$  of the projectile in the laboratory system. We illustrate the process for the case of  $\varepsilon_1$  and  $\varepsilon_2$ . In Fig. 7.1 the two situations to be compared are shown. We now form an invariant for situation (b):

$$(E_L + m_2 c^2) - c^2 p_L^2 = m_1^2 c^4 + m_2^2 c^4 + 2m_2 c^2 E_L \equiv s \tag{7.8}$$

We now calculate the same invariant using situation (a). It equals

$$(\varepsilon_1 + \varepsilon_2)^2 = 4E_0^2 = (\sqrt{m_1^2 c^4 + c^2 p_{cm}^2} + \sqrt{m_2^2 c^4 + c^2 p_{cm}^2})^2 \tag{7.9}$$

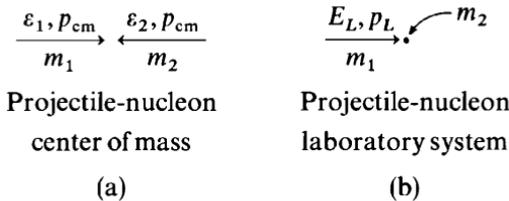


FIG. 7.1

<sup>†</sup>The Kerman et al.  $t_{KMT}$  is  $(2\pi)^{-3} t_{pn}$ . This factor is a consequence of the differing normalizations. In the KMT case the matrix elements of  $\hat{t}$  are taken with respect to plane waves  $(1/2\pi)^{3/2} e^{ik \cdot r}$ , so that (7.5) is replaced by  $t_{KMT} = -(\hbar^2 c^2 / 2\pi^2 E_0) f$ . The derivation of (7.5) is similar to that of (7.2). Terms on the order of  $(m_p c^2 - m_n c^2) / E_0$  and  $V / E_0$  are neglected. The presence of  $E_0$  in the denominator of  $f$  indicates that the assumption has been made that  $V$  transforms like the fourth component of a vector.

Equating (7.8) with (7.9) yields an equation for  $c^2 p_{cm}^2$ . This is readily solved to yield

$$p_{cm}^2 = \frac{m_2^2 c^4 p_L^2}{m_1^2 c^4 + m_2^2 c^4 + 2m_2 c^2 E_L} \sim \frac{m_2^2 c^4 p_L^2}{s}$$

or

$$p_{cm} = \frac{m_2 c^2 p_L}{2E_0} = \frac{m_2 c^2 p_L}{\sqrt{s}} \quad (7.10)$$

It then follows that

$$\begin{aligned} \varepsilon_1 &= \frac{m_2 E_L + m_1 c^2}{2E_0} = \frac{m_2 E_L + m_1 c^2}{\sqrt{s}} \\ \varepsilon_2 &= \frac{m_2 c^2 (E_L + m_2 c^2)}{2E_0} = \frac{m_2 c^2 (E_L + m_2 c^2)}{\sqrt{s}} \end{aligned} \quad (7.11)$$

A little manipulation will show that

$$\varepsilon_1 \varepsilon_2 = E_0^2 \left[ 1 - \frac{1}{(1 + m_1^2/m_2^2 + 2E_L/m_2 c^2)^2} \left( 1 - \frac{m_1^2}{m_2^2} \right)^2 \right] \quad (7.12)$$

For the situations to be considered in this volume it is a good approximation to take

$$\varepsilon_1 \varepsilon_2 \simeq E_0^2 \quad (7.13)$$

equating the arithmetic mean of  $\varepsilon_1$  and  $\varepsilon_2$  with the geometric mean. Using the exact equation (7.12) in the discussion is not difficult, but to keep the results simple we shall employ (7.13), so that

$$t_{PN} = -4\pi\hbar^2 c^2 \frac{E_0}{E_1 E_2} f \quad (7.14)$$

We are now left with the determination of  $E_1 E_2$ .  $E_0$  in terms of  $E_L$  is obtained from equating  $4 E_0^2$  to the right-hand side of (7.8). The kinematic situations involved in determining  $E_1$  and  $E_2$  are illustrated in Fig. 7.2, where  $\mathcal{E}_2$  is the energy of the target nucleus in the projectile-nucleus center-of-mass frame. We take

$$E_2 = \frac{\mathcal{E}_2}{A}$$

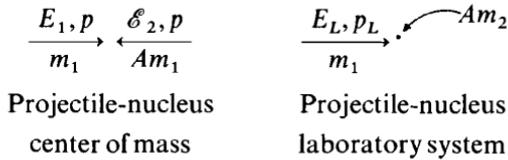


FIG. 7.2

The analysis used to obtain (7.11) can be repeated, with the result

$$E_1 E_2 = \frac{m_2 c^2 (E_L + Am_2 c^2) (Am_2 c^2 E_L + m_1^2 c^4)}{A^2 m_2^2 c^4 + m_1^2 c^4 + 2Am_2 c^2 E_L} \quad (7.15)$$

Inserting this result into (7.14) and using (7.10) yields the final result:

$$t_{PN} = -2\pi\hbar^2 c^2 \frac{k_L}{k_{cm} E_L (1 + m_1^2 c^2 / Am_2 E_L) (1 + E_L / Am_2 c^2)} f_{PN} \quad (7.16)$$

We leave it as a problem to show that this reduces correctly in the nonrelativistic limit ( $E_L = m_1 c^2$ ).

(c) *Breit Kinematics.* In employing (7.16) in the optical model Schrödinger equation (7.2), one must use values of  $f_{PN}$  that cannot be obtained from the analysis of the scattering of the projectile by a free nucleon. This point becomes clear when one examines the Lippmann–Schwinger equation, corresponding to (7.2), in momentum space. The Lippmann–Schwinger equation for the transition amplitude  $\mathcal{F}'$  is

$$\mathcal{F}' = V_{\text{opt}}^{(1)} + V_{\text{opt}}^{(1)} \frac{1}{E^{(+)} - K} \mathcal{F}'$$

where  $K$  is the kinetic energy operator. Taking plane wave matrix elements of  $\mathcal{F}'$  in the projectile–nucleus coordinate frame yields

$$\langle \mathbf{k} | \mathcal{F}' | \mathbf{k}' \rangle = \langle \mathbf{k} | V_{\text{opt}}^{(1)} | \mathbf{k}' \rangle + \int \frac{d\mathbf{k}''}{(2\pi)^3} \langle \mathbf{k} | V_{\text{opt}}^{(1)} | \mathbf{k}'' \rangle \frac{1}{E^{(+)} - E(\mathbf{k}'')} \langle \mathbf{k}'' | \mathcal{F}' | \mathbf{k}' \rangle \quad (7.17)$$

For elastic projectile–nucleus scattering, the magnitude of  $\mathbf{k}$  and  $\mathbf{k}'$  are equal and the corresponding energy  $E(k)$  is equal to  $E$ . Nonrelativistically,  $E = \hbar^2 k^2 / 2\mu$  ( $\mu =$  reduced mass). We then say that the matrix elements of  $\mathcal{F}'$  are on the energy shell if  $V_{\text{opt}}^{(1)}$ , a function of  $E$ , is also evaluated at  $E = E(k)$ . In the first-order theory  $V_{\text{opt}}^{(1)}$  depends on the  $t$  matrix for projectile–nucleon

scattering in the projectile–nucleus reference frame,  $t_{PN}$ . The requirement that  $V_{\text{opt}}^{(1)}$  is on the energy shell translates into the requirement that  $t_{PN}$  is on the energy shell and therefore can be obtained from the analysis of the experimental projectile–nucleon scattering. This last statement is valid to the extent that the kinematic regions which are allowed for projectile scattering from a free nucleon and from a nucleon embedded in the nucleus overlap. This overlap is generally not complete since the scattering from a nucleon embedded in a nucleus can involve momentum transfers  $\mathbf{q} = (\mathbf{k} - \mathbf{k}')$  which are larger than those that can occur when the target nucleon is free.

To illustrate this point, suppose that the projectile is a nucleon. Then in the nucleon–nucleon center-of-mass frame each nucleon has a momentum  $\mathbf{k}/2$  initially. The maximum momentum change occurs for  $180^\circ$  scattering, yielding a maximum value of  $q^2$  equal to  $k^2$ . In the laboratory frame (for simplicity we take the nucleus to be infinitely massive so that the projectile–nucleus frame and the laboratory frame are identical),  $q_{\text{lab}}^2$  is given by  $2k^2(1 - \cos \theta)$ , where  $\theta$  is the scattering angle. It is clear that  $q_{\text{lab}}^2$  will exceed  $k^2$ , the maximum  $q^2$  for scattering by a free nucleon, for  $\theta$  greater than  $60^\circ$ . Thus for angles greater than  $60^\circ$  it is no longer possible to obtain  $t_{PN}$  from the experimental  $t_{PN}$ .

Turning to the second term on the right-hand side of (7.17), the integral over  $\mathbf{k}''$  involves values of  $\langle \mathbf{k} | V_{\text{opt}}^{(1)} | \mathbf{k}'' \rangle$  that are not on the energy shell since  $\mathbf{k}''$  can assume any magnitude. However,  $V_{\text{opt}}^{(1)}$  involves the nuclear form factor  $\tilde{\rho}(\mathbf{q})$ , which decreases rapidly with increasing  $qR$ , where  $R$  is the nuclear radius parameter. As a consequence, the off-energy-shell contributions of  $\langle \mathbf{k} | V_{\text{opt}}^{(1)} | \mathbf{k}'' \rangle$  will be small barring a singular behavior of  $\hat{t}(\mathbf{q})$  when one deviates from on-shell kinematics.

A common method for estimating off-the-energy-shell matrix elements of  $f_{PN}$  involves establishing an analytic form for  $t_{PN}$  as a function of  $E$  and  $\mathbf{q}$  [e.g.,  $a(E)e^{-b(E)q^2}$  often used] from the on-shell experimental data. Then one substitutes in that form, that is, treats  $E$  and  $q$  as independent variables to obtain the value of  $f_{PN}$  off-shell. This procedure presumes a smooth behavior of  $f_{PN}$  as a function of these variables.

Another method which we shall now describe resolves the problem of the overlap of the kinematic regions allowed in free nucleon–projectile scattering and that allowed in nucleus–projectile scattering. We return to (4.37):

$$\tilde{V}_{\text{opt}}^{(1)}(\mathbf{k}, \mathbf{k}') = \frac{A-1}{(2\pi)^3} \int ds \int ds' \tilde{K}(\mathbf{s}, \mathbf{s}') \tilde{t}\left(\frac{\mathbf{k} + \mathbf{s}}{2}, \frac{\mathbf{k}' + \mathbf{s}'}{2}\right) \delta(\mathbf{s} - \mathbf{k} - \mathbf{s}' + \mathbf{k}') \quad (4.37)$$

where

$$\tilde{K}(\mathbf{s}, \mathbf{s}') = \int d\mathbf{r} \int d\mathbf{r}' K(\mathbf{r}, \mathbf{r}') e^{-i\mathbf{s}\cdot\mathbf{r} + i\mathbf{s}'\cdot\mathbf{r}'}$$

The variables  $\mathbf{k}$  and  $\mathbf{k}'$  multiplied by  $\hbar$  are momenta in the projectile–nucleus

frame of reference. We choose a local density approximation for  $K$ :

$$K(\mathbf{r}, \mathbf{r}') = \rho \left( \frac{\mathbf{r} + \mathbf{r}'}{2} \right) e^{i(\mathbf{k}/A)\cdot\mathbf{r} - i(\mathbf{k}'/A)\cdot\mathbf{r}'} \quad (7.18)$$

In the Fermi-gas approximation equation (6.10) this amounts to replacing the  $3j_1((x)/x)$  factor by its value at  $x=0$ . The exponential factors in (7.18) are appropriate to the nucleon-nucleus reference frame, where  $\mathbf{k}'$  is the momentum of the nucleon and  $-\mathbf{k}'$  the momentum of the nucleus. Then each nucleon in the nucleus has a momentum  $-\mathbf{k}'/A$ , neglecting the momentum of these nucleons with respect to the nuclear center of mass. Inserting (7.18) into the equation for  $\tilde{K}(\mathbf{s}, \mathbf{s}')$  and integrating yields

$$K(\mathbf{s}, \mathbf{s}') = (2\pi)^3 \delta \left( \frac{\mathbf{Q}}{A} - \frac{\mathbf{s} + \mathbf{s}'}{2} \right) \tilde{\rho} \left( \mathbf{s}' - \mathbf{s} + \frac{\mathbf{q}}{A} \right) \quad (7.19)$$

where we have introduced

$$\mathbf{Q} \equiv \frac{\mathbf{k} + \mathbf{k}'}{2} \quad \text{and} \quad \mathbf{q} = \mathbf{k} - \mathbf{k}'$$

Inserting (7.19) in (4.37) yields a factorized expression for  $\tilde{V}_{\text{opt}}^{(1)}(\mathbf{k}, \mathbf{k}')$ :

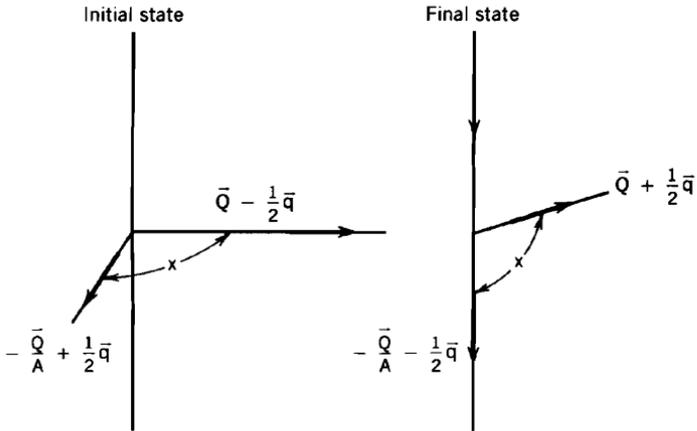
$$\tilde{V}_{\text{opt}}^{(1)}(\mathbf{k}, \mathbf{k}') = (A-1) \tilde{\rho} \left( -\mathbf{q} \left( 1 - \frac{1}{A} \right) \right) \tilde{t} \left( \frac{\mathbf{Q}(1+1/A) + \mathbf{q}}{2}, \frac{\mathbf{Q}(1+1/A) - \mathbf{q}}{2} \right) \quad (7.20)$$

In the  $Pn$  frame,  $\tilde{t}$  describes the elastic scattering of a nucleon with momentum  $\mathbf{Q} - \mathbf{q}/2$  by a target nucleon of momentum  $-\mathbf{Q}/A + \mathbf{q}/2$ , with the final momenta being given by  $\mathbf{Q} + \mathbf{q}/2$  and  $-\mathbf{Q}/A - \mathbf{q}/2$ , respectively. This is referred to as *Breit kinematics* (see Fig. 7.3). Note that for  $k = k'$ ,  $\mathbf{Q} \cdot \mathbf{q}$  equals zero, so that the energies are equal before and after the collision. Moreover, the effective kinetic energy in the laboratory frame when  $k = k'$  is given nonrelativistically by

$$T_{\text{lab}}^{(\text{eff})} = \frac{1}{2m} \left[ Q^2 \left( 1 + \frac{1}{A} \right)^2 + \mathbf{q}^2 \right] \quad (7.21)$$

In words, the two-body scattering occurs with the Breit momenta and with the effective energy given by (7.21) when the scattering is on the energy shell. This effective energy varies with the angle of scattering.

To extrapolate  $V_{\text{opt}}^{(1)}$  off the energy shell ( $k' \neq k$ ),  $\tilde{t}$  in (7.20) is replaced by  $\tilde{t}(\mathbf{q})$  evaluated at the effective laboratory energy given by (7.21), which varies with momentum transfer  $q$ . Tables of  $\tilde{t}(\mathbf{q})$  are given by McNeil, Ray, and Wallace (83). The resulting potential is nonlocal, taking into account to some extent the nonlocality of the two-body transition matrix. However, as is evident from the



**FIG. 7.3.** Two vectors representing the momenta of the incident nucleon,  $(\mathbf{Q} - \frac{1}{2}\mathbf{q})$ , and the target nucleon,  $(-\frac{1}{A}\mathbf{Q} + \frac{1}{2}\mathbf{q})$  maintain the angle between them and their magnitudes after scattering. Scattering results in a rigid rotation of the two vectors.

interpretation of (7.20) for  $V_{\text{opt}}^{(1)}$ , the momentum of the nucleons in the nucleus is neglected. An evaluation of the consequences of this treatment of off-shell effects has been investigated by Picklesimer, Tandy, Thaler, and Wolfe (84). Significant effects are obtained for nucleon projectile energies below 300 MeV.

**Note.** The relativistic generalization of (7.21) is

$$T_{\text{lab}}^{(\text{eff})} = \frac{s}{2mc^2} - 2mc^2$$

where

$$s = (E_p + E_n)^2 - \hbar^2 c^2 Q^2 \left(1 - \frac{1}{A}\right)^2$$

$$E_p^2 = \hbar^2 c^2 \left( Q^2 + \frac{q^2}{4} + \frac{m^2 c^2}{\hbar^2} \right) \quad E_n^2 = \hbar^2 c^2 \left[ \left(\frac{Q}{A}\right)^2 + \frac{q^2}{4} + \frac{m^2 c^2}{\hbar^2} \right]$$

## 8. AN EXAMPLE: PROTON-NUCLEUS SCATTERING

We conclude this chapter with a brief description of the application of multiple scattering theory to the scattering of high-energy protons by spin-zero nuclei. The objective will be to provide a qualitative understanding rather than a definitive comparison of experiment with theory. For thorough discussions the reader should consult the papers by Chaumeaux, Layly, and Schaeffer (78) and Ray (79) (see also Chapter IX).

The starting point is the nucleon-nucleon amplitude for the scattering of the incident proton by the  $i$ th nucleon of the target nucleus,  $f$ . In the nucleon-nucleon center-of-mass frame  $f_{i0}$  has the Wolfenstein form:

$$f_{i0}(\mathbf{k}_{\text{cm}}, \mathbf{k}'_{\text{cm}}) = A'_i + B'_i \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_i + C'_i (\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_i) \cdot (\mathbf{q}_{\text{cm}} \times \mathbf{Q}_{\text{cm}}) \\ + D'_i (\boldsymbol{\sigma}_0 \cdot \hat{\mathbf{Q}}_{\text{cm}}) (\boldsymbol{\sigma}_i \cdot \hat{\mathbf{Q}}_{\text{cm}}) + E'_i (\boldsymbol{\sigma}_0 \cdot \mathbf{q}_{\text{cm}}) (\boldsymbol{\sigma}_i \cdot \mathbf{q}_{\text{cm}}) \quad (8.1)$$

The variable  $\mathbf{k}'_{\text{cm}}$  is the incident momentum of the proton in the two-nucleon center-of-mass frame, and  $\mathbf{k}_{\text{cm}}$  is the final momentum. In addition,

$$\mathbf{q}_{\text{cm}} = \mathbf{k}_{\text{cm}} - \mathbf{k}'_{\text{cm}} \quad \mathbf{Q}_{\text{cm}} = \frac{1}{2}(\mathbf{k}_{\text{cm}} + \mathbf{k}'_{\text{cm}}) \quad (8.2)$$

The vector  $\hat{\mathbf{Q}}_{\text{cm}}$  is a unit vector in the direction of  $\mathbf{Q}_{\text{cm}}$ . The coefficients  $A'_i$ , and so on, in (8.1) are scalar functions of  $\mathbf{k}_{\text{cm}}$  and  $\mathbf{k}'_{\text{cm}}$ , that is, functions of  $k_{\text{cm}}^2 = k'_{\text{cm}}^2$  and  $\mathbf{k}_{\text{cm}} \cdot \mathbf{k}'_{\text{cm}}$ . They depend on isospin as follows:

$$A' = A'_+ + \tau_0 \cdot \tau_i A'_-$$

so that

$$A'_{pn} = \langle pn | A' | pn \rangle = A'_+ - A'_- \quad A'_{pp} = A'_+ + A'_- \quad (8.3)$$

The reader should verify that (8.1) is the most general form, depending at most bilinearly on  $\mathbf{q}_{\text{cm}}$  and  $\mathbf{Q}_{\text{cm}}$ , which is rotationally, time-reversal, and space-reflection invariant. (Note that under time reversal  $\mathbf{k} \rightarrow -\mathbf{k}'$ .)

To use (7.16) to obtain the value of  $t$  in the nucleon-target nucleus frame, we must transform  $\mathbf{k}_{\text{cm}}$  and  $\mathbf{k}'_{\text{cm}}$  to the projectile-nucleus frame:

$$\mathbf{q} = \mathbf{k} - \mathbf{k}' = \mathbf{q}_{\text{cm}} \quad \mathbf{Q} = \frac{1}{2}(\mathbf{k} + \mathbf{k}') \simeq \frac{k}{k_{\text{cm}}} \mathbf{Q}_{\text{cm}} \\ (\mathbf{q}_{\text{cm}} \times \hat{\mathbf{Q}}_{\text{cm}}) = \frac{1}{k} (\mathbf{q} \times \mathbf{Q})$$

The second of these equations is valid at small angles ( $\mathbf{k}' \sim \mathbf{k}$ ) only. Finally, using (7.16), one finds that

$$[\tilde{t}_{pN}(\mathbf{k}, \mathbf{k}')]_{i0} = A_i + B_i \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_i + C_i (\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_i) \cdot (\mathbf{q} \times \mathbf{Q}) \\ + D_i (\boldsymbol{\sigma}_0 \cdot \mathbf{Q}) (\boldsymbol{\sigma}_i \cdot \mathbf{Q}) + E_i (\boldsymbol{\sigma}_0 \cdot \mathbf{q}) (\boldsymbol{\sigma}_i \cdot \mathbf{q}) \quad (8.4)$$

where

$$A_i = \eta(E_L) \frac{k_L}{k_{\text{cm}}} A'_i \quad B_i = \eta(E_L) \frac{k_L}{k_{\text{cm}}} B'_i \quad C_i = \eta(E_L) \frac{k_L}{kk_{\text{cm}}} C'_i \\ D_i = \eta(E_L) \frac{k_L}{k_{\text{cm}} k^2} D'_i \quad E_i = \eta(E_L) \frac{k_L}{k_{\text{cm}}} E'_i \quad (8.5)$$

where  $\eta(E_L)$  is obtained from (7.16):

$$\eta(E_L) = -\frac{2\pi\hbar c^2}{E_L} \frac{1 + [2A/(A^2 + 1)](E_L/mc^2)}{1 + [A/(A^2 + 1)](E_L/mc^2 + mc^2/E_L)} \quad (8.6)$$

The experimental values of the coefficients in (8.1) are fitted using a Gaussian form: for example,

$$A'_{pp} = A_{pp}(0)e^{-\alpha_{pp}q^2}$$

where  $A'_{pp}(0)$  and  $\alpha_{pp}$  are complex functions of the energy.

An example of such a fit [Wallace and Alexander (80)] of the nucleon–nucleon amplitudes is given in Table 8.1 for the incident proton momentum in the laboratory frame of 1.7 GeV/c (kinetic energy 1 GeV). At that time, 1980, the only well-known proton–neutron amplitude is  $A'_{pn}$ . In the  $p$ – $p$  case the  $B'_{pp}$ ,  $D'_{pp}$  and  $E'_{pp}$  amplitudes are poorly known. Because of these uncertainties, it has been the practice in applications to multiple scattering to neglect the  $B'$ ,  $D'$ , and  $E'$  terms in (8.1) and to fit the nucleon–nucleon data with the  $A'$  and  $C'$  coefficients only. Note that the  $C'_{pn}$  term is *not* determined from nucleon–nucleon scattering but by elastic proton scattering from  $^4\text{He}$ .

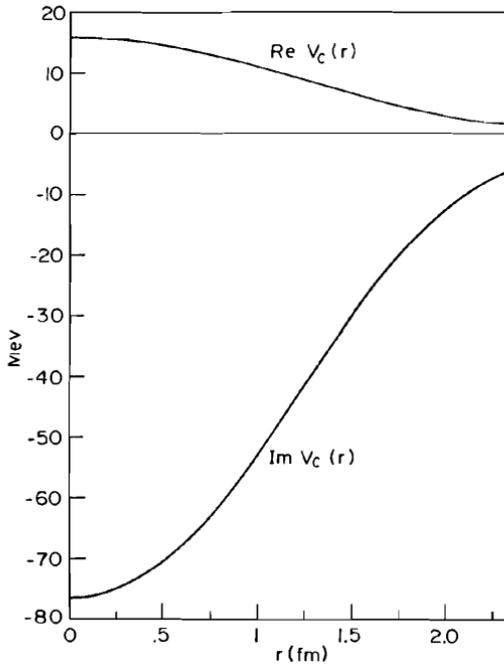
Focusing on the contributions of the  $A'$  term, one should bear in mind that  $\rho(\mathbf{q})$  varies much more rapidly than  $\tilde{t}(q)$ , so that over a considerable range the optical potential is given by  $\tilde{t}(0)\rho(\mathbf{q})$ . To illustrate, take  $\rho(\mathbf{q})$  to be a Gaussian

$$\tilde{\rho}(\mathbf{q}) = e^{-R^2q^2/6}$$

where  $R$  is the root-mean-square radius of the nucleus, which we shall take as roughly equal to the nuclear radius. Then the quantity to be compared with  $\alpha_{pp}$  or  $\alpha_{pn}$  is  $R^2/6\hbar^2c^2 \sim 6A^{2/3}(\text{GeV}/\hbar c)^{-2}$ . The latter is far greater than  $\alpha_{pp}$  or  $\alpha_{pn}$  for even light nuclei [see discussion in deShalit and Feshbach (74, 109)]. A second feature originates in the large imaginary component of  $A'$  terms. This has the consequence that the central part,  $V_c$ , of the optical potential is highly

TABLE 8.1

$A'_{pp}(0) = (-1.126 + 6.767i)(\text{GeV}/\hbar c)^{-1}$	$\alpha_{pp} = (5.08 + 0.63i)$
$A'_{pn}(0) = (-1.695 + 5.628i)(\text{GeV}/\hbar c)^{-1}$	$\alpha_{pn} = (2.93 + 0.0i)$
$B'_{pp}(0) = (-1.431 - 0.320i)(\text{GeV}/\hbar c)^{-1}$	$\beta_{pp} = (5.84 + 7.44i)$
$C'_{pp}(0) = (4.349 + 7.559i)(\text{GeV}/c)^{-2}$	$\gamma_{pp} = (3.91 + 0.596i) \times (\text{GeV}/\hbar c)^{-2}$
$C'_{pn}(0) = (-2.355 - 1.654i)(\text{GeV}/c)^{-2}$	$\gamma_{pn} = (4.00 - 2.80i)$
$D'_{pp}(0) = (.070 + 1.140i)(\text{GeV}/c)^{-1}$	$\delta_{pp} = (4.94 - 7.41i)$
$E'_{pp}(0) = (1.61.191 - 1.390i)(\text{GeV}/c)^{-3}$	$\epsilon_{pp} = (15.6 + 1.12i)$



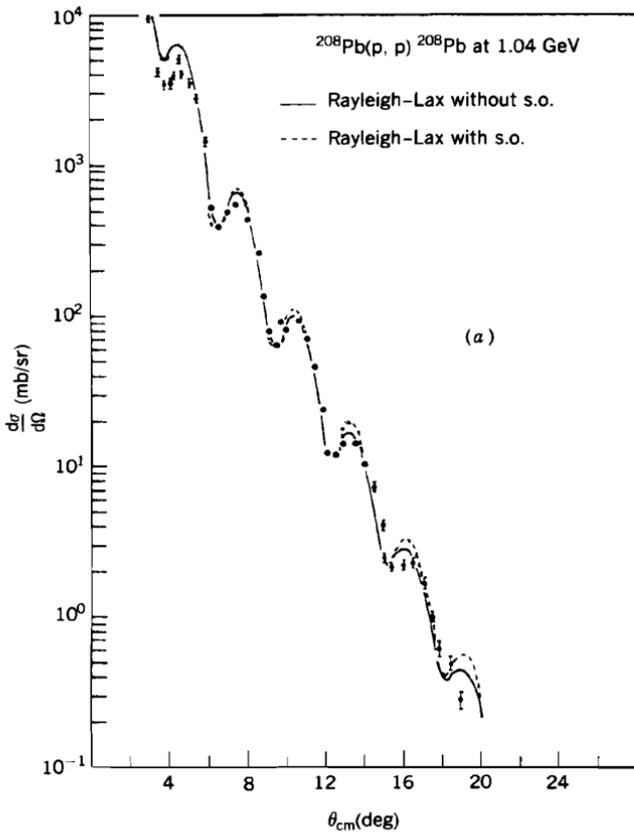
**FIG. 8.1.** Multiple scattering optical potential for the elastic scattering of 1-GeV protons by  ${}^4\text{He}$  (central potential). [From Parmentola and Feshbach (82).]

absorptive. This is illustrated in Fig. 8.1, which gives the optical potential for 1-GeV protons in  ${}^4\text{He}$ . We observe that the real part of  $V_c$  is weak and repulsive while the imaginary part is relatively large. As a consequence, the corresponding angular distribution exhibits the oscillations characteristic of Fraunhofer diffraction [ $f \sim J_1(2kR \sin(\theta/2))2kR \sin(\theta/2)$ , Eq. (5.25b)] of the incident wave by an absorbing sphere (see Fig. 8.2). The positions of the minima of these oscillations depend only on the radius of the sphere, in the case of a sphere with a well-defined sharp radius. In the more realistic case,  $\rho(r)$  will involve a radius parameter which will then determine the positions of the minima. Moreover, these are stable against the addition of spin-orbit contributions to the optical potential originating in the  $C'$  term of (8.1), of correlation effects, or of Coulomb terms, as demonstrated in Fig. 8.3. Chaumeaux et al. point out that the nucleon-nucleon amplitude can be changed by an overall phase which can be a function of  $q^2$  without disturbing the fit to experiment. Such a change will, however, have an effect on the multiple scattering since the phases of the scattering amplitudes emanating from different nucleons will be modified, thereby changing the way in which they interfere (see Fig. 8.4). Note again the stability of the position of the minima.

On the other hand, a change in the density distribution has a noticeable effect, as can be seen in Fig. 8.5, where the impact of changing the neutron density is illustrated.

From this discussion it should be clear that the scattering of protons in the range 1 GeV provides a method for the determination of the neutron density within nuclei. The proton density is taken from high-energy electron scattering after removal of the finite proton charge radius in order to obtain the point proton density. Some of the results thus obtained by Ray (79) (which include important consideration of additional electromagnetic effects of which the interested reader should be aware) are given in Table 8.2.

The uncertainty in the  $\Delta r_{np} (\equiv \langle r_n^2 \rangle^{1/2} - \langle r_p^2 \rangle^{1/2})$  is  $\pm 0.05$  fm.  $\Delta r_{np}$  is for a given nucleus, while  $\Delta r_{nn}$  is evaluated for two isotopes with differing neutron number. Examples of the deduced neutron density is shown in Fig. 8.6 for the



**FIG. 8.2.** (a) Comparison of experimental angular distribution for the elastic scattering of 1.04-GeV protons by  $^{208}\text{Pb}$  with the predictions employing the Rayleigh-Lax potential with and without spin-orbit (s.o.) terms. The density-dependent Hartree-Fock densities are used. [From Bordy and Feshbach (77).] (b) Comparison of the angular distribution for the elastic scattering of protons by  $^{16}\text{O}$ ,  $^{40}\text{Ca}$ ,  $^{58}\text{Ni}$ ,  $^{60}\text{Ni}$ ,  $^{62}\text{Ni}$ ,  $^{64}\text{Ni}$ ,  $^{90}\text{Zr}$ , and  $^{208}\text{Pb}$  with predictions employing the Rayleigh-Lax potentials. The Hartree-Fock-Bogoliubov densities are used. [From Chaumeaux, Layly, and Schaeffer (78).]

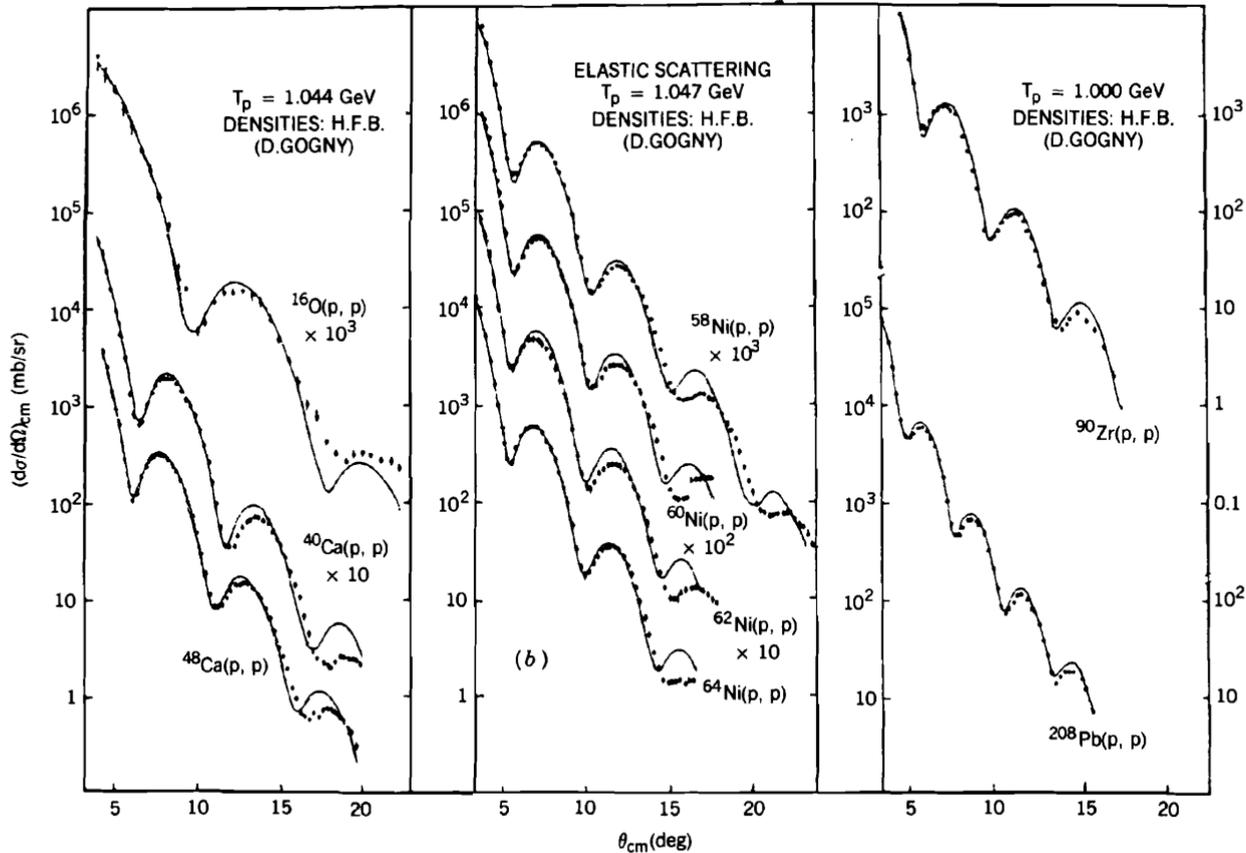
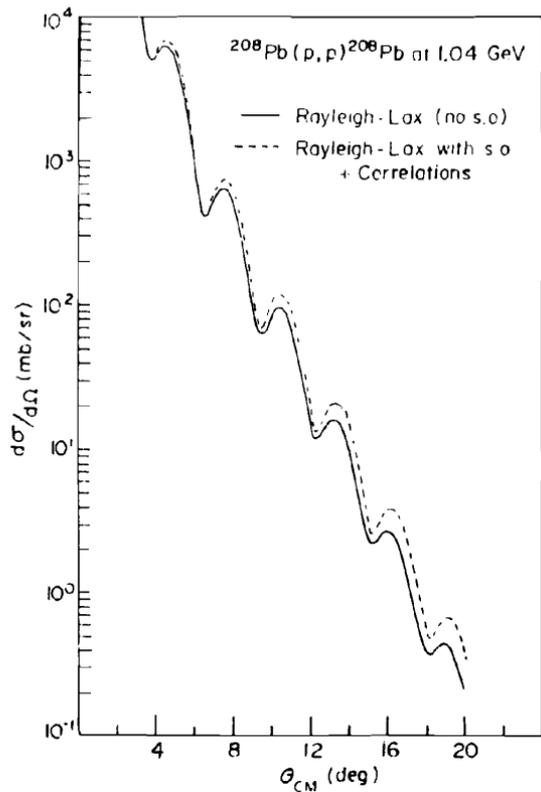
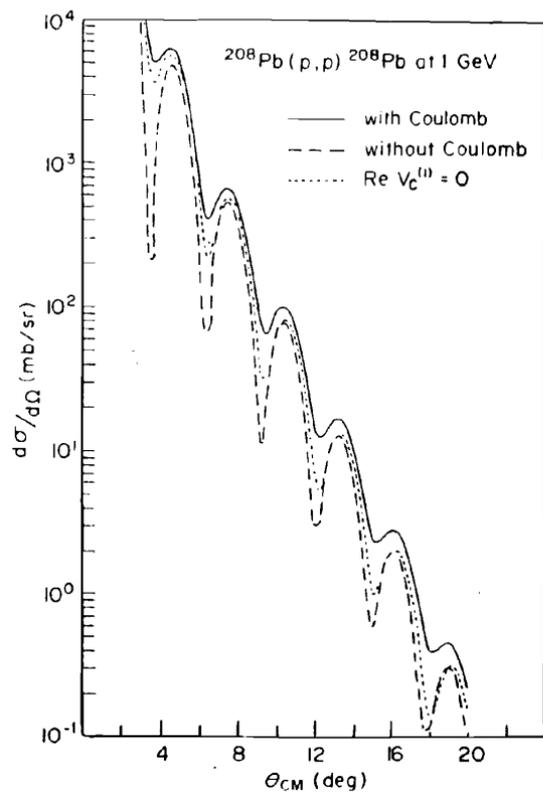


FIG. 8.2. (Continued)

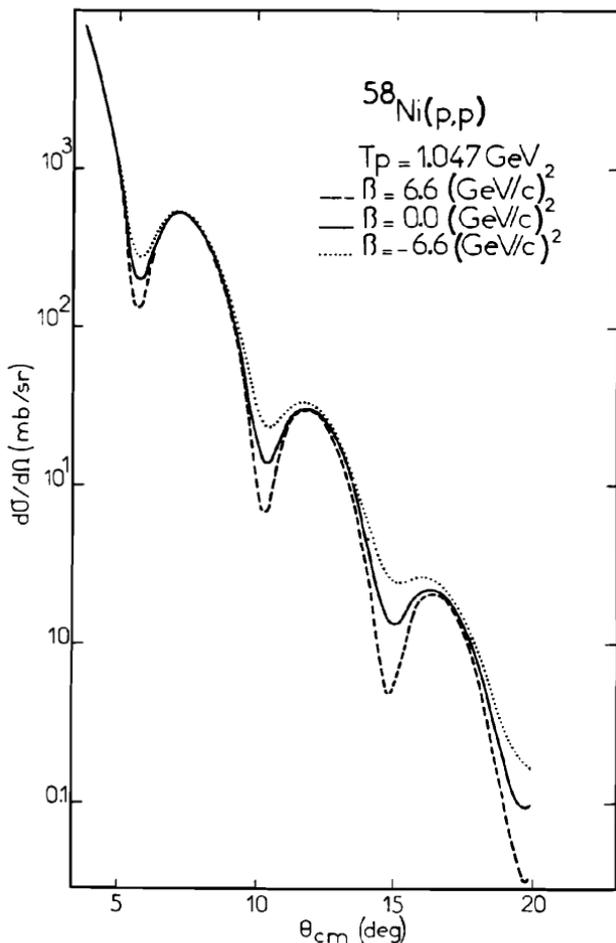


(a)



(b)

**FIG. 8.3.** (a) Effect of the spin-orbit interaction and correlations on the elastic scattering by  $^{208}\text{Pb}$ ; effect of the Coulomb interaction. [From Boridy and Feshbach (77).]



**FIG. 8.4.** Change of the cross section when the nucleon–nucleon amplitude is multiplied by a phase factor  $e^{i\beta q^2}$  for different values of  $\beta$ . [From (Chaumeaux, Layly, and Schaeffer (78).)]

Ni isotopes. From Table 8.2 we see reasonable agreement with Negele's density-dependent Hartree–Fock calculation, an agreement that is also present in Fig. 8.6. The outstanding major disagreement is in the value of  $\Delta r_{nn}$  for the pair  $^{48}\text{Ca}$  and  $^{40}\text{Ca}$ .

Both Chaumeaux et al. and Ray take correlations into account. According to both authors (see Appendix B at the end of this chapter), the effects are appreciable at the larger angles. However, they are of the same order as effects arising in the uncertainties in the input data and small electromagnetic effects involving the form factor of the neutron. The effect of spin-dependent correlations, including those arising from  $B'$ ,  $C'$ ,  $D'$ , and  $E'$  terms, on elastic scattering have been investigated by Lambert and Feshbach (73) and Parmentola

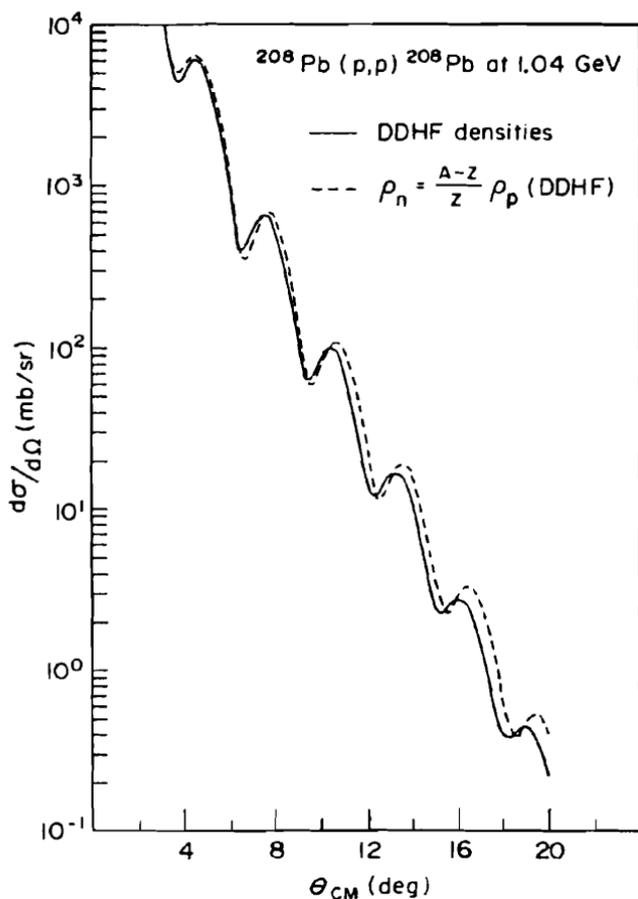
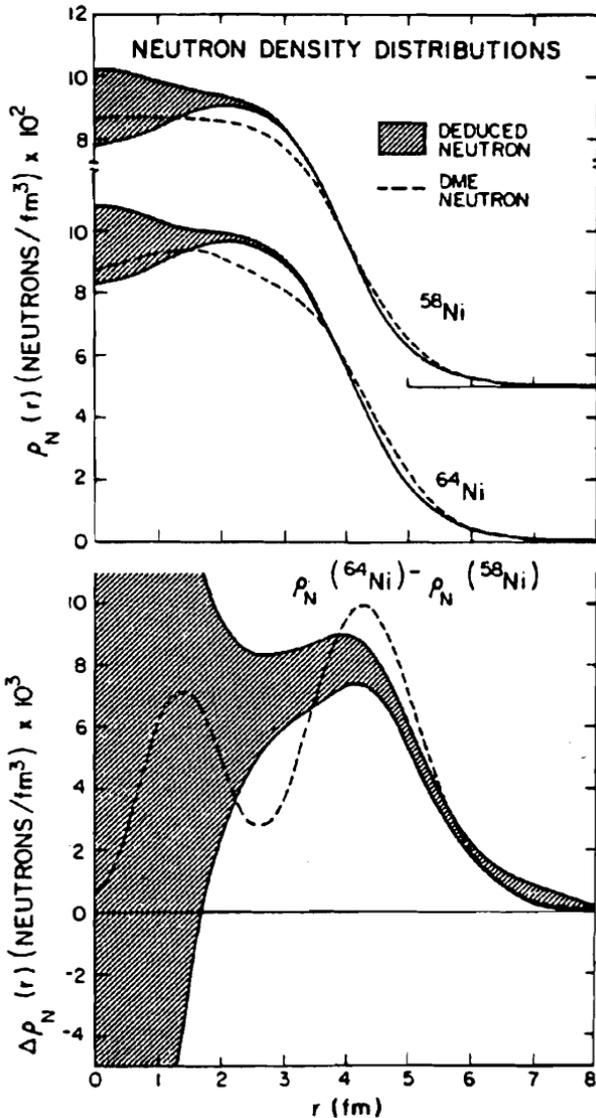


FIG. 8.5. Effect of changing the neutron density distribution. [From Boridy and Feshbach (77).]

TABLE 8.2

Nucleus	$\langle r_n^2 \rangle^{1/2}$ (fm)	$\langle r_p^2 \rangle^{1/2}$ (fm)	$\langle r_c^2 \rangle^{1/2}$ (fm)	$\Delta r_{np}$	$\Delta r_{nn'}$	DDHF	
						$\Delta r_{np}$	$\Delta r_{nn'}$
$^{40}\text{Ca}$	3.491	3.392	3.482	0.10		-0.05	
$^{48}\text{Ca}$	3.625	3.396	3.470	0.23	$0.13 \pm 0.04$	0.19	0.26
$^{58}\text{Ni}$	3.700	3.686	3.772	0.01		0.00	
$^{64}\text{Ni}$	3.912	3.745	3.845	0.17	$0.21 \pm 0.02$	0.13	0.18
$^{116}\text{Sn}$	4.692	4.546	4.619	0.15		0.12	
$^{124}\text{Sn}$	4.851	4.599	4.670	0.25	$0.16 \pm 0.02$	0.21	0.13
$^{208}\text{Pb}$	5.611	5.453	5.503	0.16		0.20	



**FIG. 8.6.** Point neutron density distributions for  $^{58,64}\text{Ni}$  deduced from second-order KMT analysis (shaded bands) and predicted by the density matrix expansion (DME) approach to Hartree-Fock theory (dashed curves). The difference between the  $^{64}\text{Ni}$ - and  $^{58}\text{Ni}$ -deduced neutron densities is compared with the DME prediction in the lower half of the figure. [From Ray (79).]

and Feshbach (82) for a  $^4\text{He}$  target. These authors find that generally the influence of the  $B$ ,  $D$ , and  $E$  terms is small if the nucleon–nucleon parameters of Table 8.1 are used. Chaumeaux et al. examine the spin-dependent effects of  $B$ ,  $C$ , and  $E$  ( $D$  is not included) in *inelastic* scattering, pointing out quite correctly the sensitivity of the angular distribution and of the polarization to these coefficients. The results for these quantities, using coefficients  $A$ ,  $B$ ,  $C$ , and  $E$  quite similar to those given in Table 8.1, are less than satisfactory. But adjustments within the uncertainties can improve comparison with experiment.

## APPENDIX A

In this section we apply the *optimal approximation method* of Gurvitz, Dedonder, and Amado (79) to projectile–nucleus scattering. We focus on the relationship between  $t_i$  and  $\tau_i$  as given in (4.19). In the course of that derivation  $\alpha_0 (\equiv E - K_0)$  of (4.4) is replaced by  $\alpha (\equiv E - K_0 - H_N)$ , as in (4.18). The method of Gurvitz et al. allows for an adjustment of the values of  $E$  so as to reduce the consequent error. Let

$$\bar{\alpha}_0 \equiv \varepsilon - K_0$$

and

$$\bar{t}_i = v_i + v_i \frac{1}{\bar{\alpha}_0} \bar{t}_i \quad (\text{A.1})$$

The equation for  $\tau_i$ , (4.7), remains unchanged:

$$\tau_i = v_i \mathcal{A} + v_i \frac{\mathcal{A}}{\alpha} \tau_i \quad (\text{4.7})$$

The dependence of both  $\bar{t}_i$  and  $\tau_i$  on  $v_i$  can be used to eliminate  $v_i$ , producing a relation between  $\bar{t}_i$  and  $\tau_i$ :

$$\tau_i = \bar{t}_i \mathcal{A} + \bar{t}_i \left( \frac{\mathcal{A}}{\alpha} - \frac{1}{\bar{\alpha}_0} \right) \tau_i \quad (\text{A.2})$$

The optimal approximation will be applied to an auxiliary quantity  $\tau'_i$  defined by

$$\tau'_i = \bar{t}_i + \bar{t}_i \left( \frac{1}{\alpha} - \frac{1}{\bar{\alpha}_0} \right) \tau'_i \quad (\text{A.3})$$

in terms of which

$$\tau_i = \tau'_i \mathcal{A} + \tau'_i \frac{\mathcal{A} - 1}{\alpha} \tau_i \quad (\text{A.4})$$

To second order,

$$\begin{aligned}
 \tau'_i &= \bar{\tau}_i + \bar{\tau}_i \frac{1}{\bar{\alpha}_0} (\bar{\alpha}_0 - \alpha) \frac{1}{\alpha} \tau'_i \\
 &\simeq \bar{\tau}_i + \bar{\tau}_i \frac{1}{\bar{\alpha}_0} (\bar{\alpha}_0 - \alpha) \frac{1}{\bar{\alpha}_0} \bar{\tau}_i \\
 &\simeq \bar{\tau}_i + \Delta\tau'_i
 \end{aligned} \tag{A.5}$$

We shall now show that it is possible to choose  $\bar{\alpha}_0$  (by choosing  $\varepsilon$ ) in a manner independent of the target nucleus, so that (A.1) describes two-body scattering and so that

$$\tau'_i = \bar{\tau}_i \tag{A.6}$$

and therefore in agreement with (4.26), with  $t_i$  replaced by  $\bar{\tau}_i$ .

To demonstrate (A.6), consider the matrix element of the second term in (A.5) ( $\equiv \Delta\tau'_i$ ) with respect to the target nucleus and the incident and emergent projectile. Let the latter have momenta  $\mathbf{p}$  and  $\mathbf{p}'$ , respectively, while the target nucleus wave function in the momentum representation is  $\psi_0(\mathbf{P}_1 - \mathbf{p}/A, \mathbf{P}_2 - \mathbf{p}/A, \dots)$ , where  $\Sigma \mathbf{P}_i = \mathbf{P}$  is the total momentum of the projectile plus target. The propagator  $(1/\bar{\alpha}_0)$  is taken to be diagonal in momentum space:

$$\left\langle \mathbf{p}', \mathbf{P}'_1, \mathbf{P}'_2 \dots \left| \frac{1}{\bar{\alpha}_0} \right| \mathbf{p}, \mathbf{P}_1, \mathbf{P}_2 \dots \right\rangle = \frac{\delta(\mathbf{p}' - \mathbf{p}) \delta(\mathbf{P}'_1 - \mathbf{P}_1) \delta(\mathbf{P}'_2 - \mathbf{P}_2) \dots}{\varepsilon - p^2/2\mu} \tag{A.7}$$

As a consequence,  $\bar{\tau}_i$  of (A.1) is diagonal in the target nucleon momenta, as  $v_i$  can only change the projectile momenta.

$$\langle \mathbf{p}', \mathbf{P}'_1, \mathbf{P}'_2 \dots | \bar{\tau}_i | \mathbf{p}, \mathbf{P}_1, \mathbf{P}_2 \dots \rangle = \langle \mathbf{p}' | \bar{\tau}_i | \mathbf{p} \rangle \delta(\mathbf{P}'_1 - \mathbf{P}_1) \delta(\mathbf{P}'_2 - \mathbf{P}_2) \dots \tag{A.8}$$

We can now proceed to evaluate the matrix element of  $\Delta\tau'_i$  for elastic scattering; the incident projectile has a momentum  $\mathbf{p}$ , and the target nucleus  $\mathbf{P} - \mathbf{p}$ . One obtains

$$\begin{aligned}
 &\langle \mathbf{p}'; \psi_0 | \Delta\tau'_i | \mathbf{p}, \psi_0 \rangle \\
 &= \int d\mathbf{p}_1 d\mathbf{p}_2 \int d\mathbf{P}_1 \int d\mathbf{P}'_1 \dots \psi_0^* \left( \mathbf{P}'_1 - \frac{\mathbf{p}'}{A}, \mathbf{P}'_2 - \frac{\mathbf{p}'}{A} \dots \right) \\
 &\quad \times \langle \mathbf{p}' | \bar{\tau}_i | \mathbf{p}_1 \rangle \frac{1}{\varepsilon_1 - p_1^2/2\mu} \langle \mathbf{p}_1, \mathbf{P}'_1, \mathbf{P}'_2, \dots | \bar{\alpha}_0 - \alpha | \mathbf{p}_2, \mathbf{P}_1, \mathbf{P}_2, \dots \rangle \\
 &\quad \times \langle \mathbf{p}_2 | \bar{\tau}_i | \mathbf{p} \rangle \frac{1}{\varepsilon_2 - p_2^2/2\mu} \psi_0 \left( \mathbf{P}_1 - \frac{\mathbf{p}}{A}, \mathbf{P}_2 - \frac{\mathbf{p}}{A}, \dots \right)
 \end{aligned} \tag{A.9}$$

Evaluating the matrix element of  $\bar{\alpha}_0 - \alpha$  gives

$$\begin{aligned}
 & \langle \mathbf{p}_1, \mathbf{P}'_1, \mathbf{P}'_2, \dots | \bar{\alpha}_0 - \alpha | \mathbf{p}_2, \mathbf{P}_1, \mathbf{P}_2, \dots \rangle \\
 &= \delta(\mathbf{p}_1 - \mathbf{p}_2) \left\{ \left[ \varepsilon_1 - \frac{p_1^2}{2\mu} - \left( E - \frac{p_1^2}{2\mu} - \sum \frac{1}{2m} \left( \mathbf{P}'_n - \frac{\mathbf{P}_1}{A} \right)^2 \right) \right] \right. \\
 & \quad \times \delta(\mathbf{P}'_1 - \mathbf{P}_1) \delta(\mathbf{P}'_2 - \mathbf{P}_2) \cdots + \sum_n V(\mathbf{P}'_n - \mathbf{P}_n) \delta(\mathbf{P}'_1 - \mathbf{P}_1) \cdots \\
 & \quad \left. \times \delta(\mathbf{P}'_{n-1} - \mathbf{P}_{n-1}) \delta(\mathbf{P}'_{n+1} - \mathbf{P}_{n+1}) \delta(\mathbf{P}'_{n+2} - \mathbf{P}_{n+2}) \cdots \right\} \quad (\text{A.10})
 \end{aligned}$$

where the assumption has been made that the nucleons in the target nucleus move in a mean field described by a local potential  $V$ . Using the Schrödinger equation satisfied by  $\psi_0(\mathbf{P}_1 - \mathbf{p}/A, \dots)$ , the integration over the  $\mathbf{P}_n$  and  $\mathbf{p}_2$  can be performed to yield

$$\begin{aligned}
 & \langle \mathbf{p}', \psi_0 | \Delta\tau_i | \mathbf{p}, \psi_0 \rangle \\
 &= \int d\mathbf{p}_1 \int d\mathbf{P}'_1 \int d\mathbf{P}'_2 \cdots \psi_0^* \left( \mathbf{P}'_1 - \frac{\mathbf{p}'}{A}, \mathbf{P}'_2 - \frac{\mathbf{p}'}{A}, \dots \right) \langle \mathbf{p}' | \tilde{t} | \mathbf{p}_1 \rangle \\
 & \quad \times \frac{1}{\varepsilon_1 - p_1^2/2\mu} \left[ \varepsilon_1 - E + \sum \frac{1}{2m} \left\{ \left( \mathbf{P}'_n - \frac{\mathbf{P}_1}{A} \right)^2 - \left( \mathbf{P}'_n - \frac{\mathbf{p}}{A} \right)^2 \right\} \right] \\
 & \quad \times \langle \mathbf{p}_1 | \tilde{t} | \mathbf{p} \rangle \frac{1}{\varepsilon_1 - p_1^2/2\mu} \psi_0 \left( \mathbf{P}'_1 - \frac{\mathbf{p}}{A}, \mathbf{P}'_2 - \frac{\mathbf{p}}{A}, \dots \right) \quad (\text{A.11})
 \end{aligned}$$

The quantity in the braces becomes  $(2/A)(\mathbf{p} - \mathbf{p}_1) \cdot \mathbf{P}'_n + (1/A^2)(\mathbf{p}_1^2 - \mathbf{p}^2)$ . At this point the *Breit coordinates* are introduced:

$$\mathbf{q} = \mathbf{p} - \mathbf{p}' \quad \mathbf{K} = \frac{1}{2}(\mathbf{p} + \mathbf{p}') \quad \mathbf{K} \cdot \mathbf{q} = 0, \quad \frac{1}{2}(p^2 + p'^2) = K^2 + \frac{1}{4}q^2 \quad (\text{A.12})$$

yielding

$$\begin{aligned}
 2(\mathbf{p} - \mathbf{p}_1) \cdot \mathbf{P}'_n &= 2(\mathbf{K} + \frac{1}{2}\mathbf{q} - \mathbf{p}_1) \cdot \mathbf{P}'_n \\
 &= 2(\mathbf{K} + \frac{1}{2}\mathbf{q} - \mathbf{p}_1) \cdot (\mathbf{P}'_n - \mathbf{K}) + 2(\mathbf{K} + \frac{1}{2}\mathbf{q} - \mathbf{p}_1) \cdot (\mathbf{K})
 \end{aligned}$$

By using time-reversal invariance of the integrand of (A.11) it can be shown that the first term of the equation above integrates to zero, so that no dependence on the coordinates  $\mathbf{P}'_n$  remains. As a consequence, the term in brackets in (A.11) becomes

$$\varepsilon_1 - E - \frac{q^2}{8mA} + \frac{(\mathbf{p}_1 - \mathbf{K})^2}{2mA}$$

It follows from (A.11) that

$$\langle \mathbf{p}'\psi_0 | \Delta\tau_i | \mathbf{p}\psi_0 \rangle = 0$$

if one chooses

$$\varepsilon_1 = E + \frac{q^2}{8mA} - \frac{(\mathbf{p}_1 - \mathbf{K})^2}{2mA} \quad (\text{A.13})$$

where  $E = p^2/2\mu$ . This choice of  $\varepsilon_1$  is independent of the target nucleon variables  $\mathbf{P}_n$ , so that as implied by (A.7) and (A.8),  $\bar{t}_i$  is the transition amplitude for the scattering following from the Schrödinger equation:

$$\left[ \frac{p^2}{2\mu} + \frac{q^2}{8mA} - \frac{(\mathbf{p}_1 - \mathbf{K})^2}{2mA} - v - \frac{p_1^2}{2\mu} \right] \phi(p_1) = 0$$

The effective energy is obtained by dropping  $v$  from this equation. One can then rewrite the term in brackets as follows:

$$\frac{p^2}{2\mu} + \frac{q^2}{8mA} - \frac{K^2}{2(mA + \mu)} - \frac{1}{2} \left( \frac{1}{\mu} + \frac{1}{mA} \right) \left( \mathbf{p}_1 - \frac{\mu}{mA + \mu} \mathbf{K} \right)^2$$

so that the effective energy is

$$E_{\text{eff}} = \frac{p^2}{2\mu} + \frac{q^2}{8mA} - \frac{K^2}{2(mA + \mu)} \quad (\text{A.14})$$

and the effective momentum operator is  $\mathbf{p}_1 - [\mu/(mA + \mu)]\mathbf{K}$ , so that the projectile momentum  $\mathbf{p}$  is replaced in the Schrödinger equation by  $\mathbf{p} - [\mu/(mA + \mu)]\mathbf{K}$ .

The development in Section 4 uses  $t_i$  rather than  $\bar{t}_i$ . In first order these two quantities will be approximately equal if the last two terms in (A.14) are small compared to the first. These ratios are on the order of  $\mu/mA$ , which is appreciable only for the lightest nuclei.

The optimal approximation reduces the second-order term to zero through the choice of  $\varepsilon_1$  given by (A.13). The question remains as to whether higher-order terms are significant. The third-order term is discussed by Gurvitz, Dedonder, and Amado (79), who conclude that it is not if  $t$  varies slowly with the energy.

## APPENDIX B CORRELATIONS

The second-order optical model potential  $V_{\text{opt}}^{(2)}$  given by (4.41) is nonlocal, with the consequence that its effect is difficult to evaluate analytically or even

numerically. One approximate method derived by Feshbach, Gal, and Hufner (71) [see also Feshbach (81) as well as papers written with Lambert (73), Ullo (74), and Parmentola (82)] replaces the Schrödinger equation containing  $V_{\text{opt}}^{(2)}$  with a pair of coupled equations. This procedure is in any event necessary if the spin-dependent terms are to be taken into account accurately as described in the text (see p. 100). In this section further approximations valid at sufficiently high projectile energies and small momentum transfers are made which yield a local form for  $V_{\text{opt}}^{(2)}$  akin to that obtained by Chaumeaux, Layly, and Schaeffer (78).

We repeat (4.41) with  $\langle \mathbf{r} | 1/\bar{\alpha} | \mathbf{r}' \rangle \equiv G(\mathbf{r}, \mathbf{r}')$ :

$$\int V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}') d\mathbf{r}' = (A-1)^2 \int d\mathbf{r}' G(\mathbf{r}_1, \mathbf{r}') \iint d\mathbf{r}_1 d\mathbf{r}_2 t(\mathbf{r}-\mathbf{r}_1)t(\mathbf{r}'-\mathbf{r}_2)C(\mathbf{r}_1, \mathbf{r}_2)\psi(\mathbf{r}') \quad (\text{B.1})$$

We now assume that

$$C(\mathbf{r}_1, \mathbf{r}_2) = \rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\gamma(\mathbf{r}_1 - \mathbf{r}_2) \quad (\text{B.2})$$

Second, we note that  $t(\mathbf{r}-\mathbf{r}_1)$  is sharply peaked at  $\mathbf{r} \sim \mathbf{r}_1$ . Assuming that  $\rho(\mathbf{r})$  varies slowly over that peak, little error is made if  $\rho(\mathbf{r}_1)t(\mathbf{r}-\mathbf{r}_1)$  is replaced by  $\rho(\mathbf{r})t(\mathbf{r}-\mathbf{r}_1)$ . Equation (B.1) then becomes

$$\int V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}') d\mathbf{r}' = (A-1)\rho(\mathbf{r}) \int d\mathbf{r}' G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')F(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}') \quad (\text{B.3})$$

where

$$F(\mathbf{r}-\mathbf{r}') \equiv \int d\mathbf{r}_1 \int d\mathbf{r}_2 t(\mathbf{r}-\mathbf{r}_1)t(\mathbf{r}'-\mathbf{r}_2)\gamma(\mathbf{r}_1 - \mathbf{r}_2) \quad (\text{B.4})$$

Note that one can show that the integral on the right-hand side of this equation is a function of  $(\mathbf{r}-\mathbf{r}')$ . Since the integrands in (B.4) consist of three sharply peaked functions,  $F$  itself is sharply peaked at  $\mathbf{r} = \mathbf{r}'$ , so that  $\rho(\mathbf{r}')$  in (B.3) can be replaced by  $\rho(\mathbf{r})$ :

$$\int V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}') d\mathbf{r}' = (A-1)^2\rho^2(\mathbf{r}) \int d\mathbf{r}' G(\mathbf{r}, \mathbf{r}')F(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}') \quad (\text{B.5})$$

This result could also have been obtained by using the local density approximation for second-order multiple scattering in nuclear matter. In that case it is exact, and in the case of a finite nucleus,  $G(\mathbf{r}, \mathbf{r}')$  is to a good approximation  $G(\mathbf{r}-\mathbf{r}')$ , so that

$$\int V_{\text{opt}}^{(2)}(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}') d\mathbf{r}' = (A-1)^2\rho^2(\mathbf{r}) \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')F(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')$$

The integral on the left-hand side can now be reduced to a local form using the Perey–Saxon (64) approximation, which is discussed in detail in Chapter V (p. 000). In the present case

$$\begin{aligned} \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') F(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}') &= \int d\mathbf{s} G(\mathbf{s}) F(\mathbf{s}) \psi(\mathbf{r} - \mathbf{s}) \\ &= \int d\mathbf{s} G(\mathbf{s}) F(\mathbf{s}) e^{-\mathbf{s} \cdot \nabla_{\mathbf{r}}} \psi(\mathbf{r}) \end{aligned}$$

so that  $V_{\text{opt}}^{(2)}(\mathbf{r}, \mathbf{r}')$  can be replaced by

$$V_{\text{opt}}^{(2)} = (A - 1)^2 \rho^2 \int d\mathbf{s} G(\mathbf{s}) F(\mathbf{s}) e^{-\mathbf{s} \cdot \nabla_{\mathbf{r}}} \quad (\text{B.6})$$

The Perey–Saxon approximation exploits the fact that the dominant component of  $\psi(\mathbf{r})$  is the plane wave  $e^{i\mathbf{k} \cdot \mathbf{r}}$ . Then

$$V_{\text{opt}}^{(2)}(\mathbf{r}) \simeq (A - 1)^2 \rho^2(\mathbf{r}) \int d\mathbf{s} G(\mathbf{s}) F(\mathbf{s}) e^{-i\mathbf{k} \cdot \mathbf{s}} \quad (\text{B.7})$$

In this approximation the  $V_{\text{opt}}^{(2)}$  dependence on  $\mathbf{r}$  is given by  $\rho^2$ , in comparison with  $V^{(1)}$ , whose spatial dependence is determined by  $\rho(\mathbf{r})$ . An improved approximation for  $V^{(2)}$ , which may be needed when the nuclear surface plays a significant role, can be obtained by Taylor expansion of  $\rho(\mathbf{r}_1)$  and  $\rho(\mathbf{r}_2)$ ; for example,  $\rho(\mathbf{r}_1)$  would be placed equal to  $\rho(\mathbf{r}) + (\mathbf{r}_1 - \mathbf{r}) \cdot \nabla \rho(\mathbf{r}) + \dots$ . Such corrections may be of importance in dealing with inelastic scattering.

Evaluation of the integral in (B.7) can be simplified by using the eikonal approximation for  $G$ . In zeroth order we neglect the excitation energy  $\bar{\epsilon}$  and the potential energy term  $V^{(1)}$ :

$$G(\mathbf{s}) = G(\mathbf{b}, \zeta) \simeq G_{\text{eik}}^{(0)} = -\frac{i\alpha}{\hbar^2 c^2 k} e^{i\mathbf{k}\zeta} \theta(\zeta) \delta(\mathbf{b}) \quad (\text{B.8})$$

where from (7.2),

$$k^2 = \frac{1}{\hbar^2 c^2} \frac{\mathcal{E}^2 - m_p^2 c^4}{1 + \mathcal{E}/A m c^2} \quad \text{and} \quad \alpha = \frac{\mathcal{E} + m_p c^2}{1 + \mathcal{E}/A m c^2} \quad (\text{B.9})$$

The energy  $\mathcal{E}$  is the energy of the system, including the rest-mass energy of the projectile but not that of the target nucleus. The function  $\theta(\zeta)$  is the unit function

$$\theta(\zeta) = \begin{cases} 1 & \zeta > 0 \\ 0 & \zeta < 0 \end{cases}$$

We must still fix the  $\zeta$  direction. This we take to be the direction of  $\mathbf{k}$ . One now obtains

$$\int d\mathbf{s} G(\mathbf{s}) F(\mathbf{s}) e^{-i\mathbf{k}\cdot\mathbf{s}} = -\frac{i\alpha}{\hbar^2 c^2 k} \int d\zeta F(0, \zeta) \theta(\zeta) = -\frac{i\alpha}{2\hbar^2 c^2 k} \int_{-\infty}^{\infty} d\zeta F(0, \zeta)$$

where it has been assumed, as is usually the case, that  $F(0, \zeta)$  is even in  $\zeta$ . Recalling the definition of  $F$ , (B.4), the integral of  $F(0, \zeta)$  can be expressed in terms of the Fourier transforms of  $t$  and  $\gamma$ :

$$\int F(0, \zeta) d\zeta = \left(\frac{1}{2\pi}\right)^2 \int d^2 q_{\perp} \tilde{t}(\mathbf{q}_{\perp}, 0) \tilde{t}(-\mathbf{q}_{\perp}, 0) \tilde{\gamma}(\mathbf{q}_{\perp}, 0) \quad (\text{B.10})$$

We parameterize this result by

$$\int F(0, \zeta) d\zeta = -(\tilde{t}(0))^2 l \quad (\text{B.11})$$

where  $l$  is a length. The final result for  $V_{\text{opt}}^{(2)}$  is

$$V_{\text{opt}}^{(2)} = \frac{i\alpha l}{2\hbar^2 c^2 k} [(A - 1)\rho(\mathbf{r})\tilde{t}(0)]^2 \quad (\text{B.12})$$

This potential is proportional to the square of  $V_{\text{opt}}^{(1)}$ , (4.40), when the short-range approximation is made for  $t(\mathbf{r} - \mathbf{r}_1)$ . A rough estimate for  $l$  is obtained by assuming the Gaussian form (see Table 8.1) for  $\tilde{t}(\mathbf{q})$ ,  $\tilde{t}(0)e^{-(\beta^2/2)q^2}$ , and for  $\gamma(\mathbf{r})$ :

$$\gamma(\mathbf{r}) = \left(\frac{r^2}{3r_c^2} - 1\right) e^{-r^2/2r_c^2} \quad (\text{B.13})$$

the form used by Chaumeaux et al. This choice for  $\gamma(\mathbf{r})$  satisfies the condition that its volume integral is zero, satisfying (2.21) when  $\rho$  can be taken to be a constant. Finally,  $C(\mathbf{r})$  approaches  $-1$  as  $r$  goes to zero. With these choices  $l$  becomes

$$l = \frac{\sqrt{2\pi}}{3} r_c \frac{1}{(1 + 2\beta^2/r_c^2)^2} \quad (\text{B.14})$$

As pointed out by Feshbach Gal, and Hüfner (71),  $l$  is reduced because of the presence of the range of the projectile-nucleon potential given by  $\beta$  in (B.14). Inserting typical values for  $r_c$  ( $\sim 0.7$  fm),  $\beta^2 = 0.4$  fm<sup>2</sup>, one finds that  $l = 0.084$  and that the magnitude of  $V_{\text{opt}}^{(2)}$  is small compared to  $V_{\text{opt}}^{(1)}$ . However, because of the dependence of  $\rho^2$  rather than  $\rho$ , there is a greater proportion of high-momentum components which can affect the cross section for large momentum transfers.