CHAPTER III

FORMAL THEORY OF NUCLEAR REACTIONS

1. INTRODUCTION

A formal theory of nuclear reactions should provide a framework within which it is possible to describe the wide range of reaction mechanisms exhibited in nuclear collisions. It should develop, directly from the nuclear Hamiltonian, the amplitude for the rapid processes, such as the single-step direct reactions as well as for the relatively slow compound nuclear resonance reactions. It should include the intermediate structure doorway state reactions and go beyond to the multistep reactions of both the direct and compound variety. It should be possible to obtain in the high-energy limit the multiple scattering approximation of Chapter II. Finally, it should permit the application of statistical considerations and thus obtain the statistical theories of nuclear reactions applicable to the various experimental situations. It should be emphasized that the formal theory only develops a framework; a framework that provides a means for inserting the physics of the reaction under consideration. Once this is done, the theory should yield expressions that allow a direct interpretation of the experimental data in terms of well-defined parameters.

As we have emphasized, reactions can be ordered according to the time delay they involve. Time delay in a given reaction can be introduced by providing alternative mechanisms to the direct one by means of which the system can proceed to the exit channel of interest. Instead of the system proceeding in a single step to the final state, it can make a transition to another channel (or channels) and so delay the development of the final state. This possibility is shown schematically in Fig. 1.1 for the case of elastic scattering for simplicity; that is, it is assumed that the energy is so low that this is the only reaction that

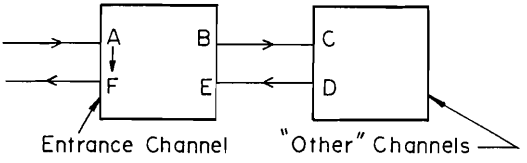

FIG. 1.1

can occur. Following the direct route the system would never leave the entrance channel. This route is symbolized by AF. But as is indicated by the figure, the time-delayed routes $ABCDEF$ can also contribute to the process. The time delay depends on the ratio of the probability of the transition from the entrance channel to the "other" channels, BC in the figure, to the probability that the system proceeds immediately to the final state, symbolized by F. However, this branching ratio is not the only parameter of importance. For example, if it were large, not only would the transition from the entrance channel be rapid but the transition back to the entrance channel would be equally quick. There would be some time delay but not necessarily a substantial one. For the latter to occur, another condition needs to be satisfied, one that inhibits the return transition.

To see what this condition is, it is necessary to decompose the other channels into a group that connect directly with the entrance channel and the remainder, as illustrated by Fig. 1.2. As indicated, the system can proceed to the doorway channels, as those channels that couple directly to the entrance channel are designated, and either proceed on to the "remaining" channels or return back to the entrance channel. The remaining channels, by definition, do not couple directly to the entrance channel. Thus a second important parameter is the ratio of the probability for the transition from the doorway to the remaining channels to the probability for the transition from the doorway back to the entrance channels. If this ratio is large, the time delay will be large for then the system will be trapped, spending a considerable fraction of the interaction time oscillating between the doorway channels and the remaining channels.

This time delay can be especially long at particular energies, the resonance energies, as can be seen from the following discussion. Suppose for a moment that the transition probability from the doorway channels to the entrance channel were zero. Then the system, if placed in other channels (consisting of

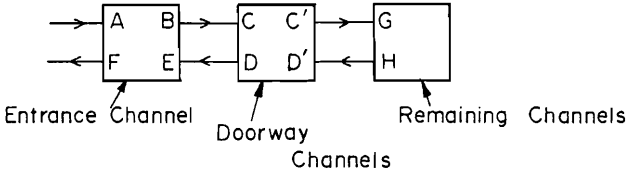

FIG. 1.2

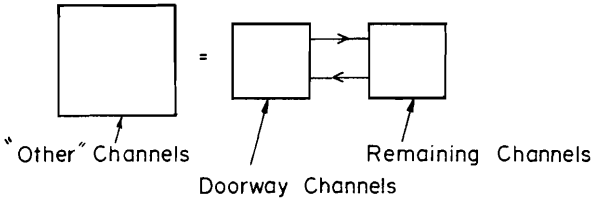

FIG. 1.3

the doorway and remaining channels), would be bound. Under these circumstances the system will oscillate between the doorway and remaining channels indefinitely but *only* at certain energies, the bound-state energies, E_b. Suppose now that the transition probability to and from the entrance channel is increased from zero. It follows that when the incident energy is correspondingly close to the binding energies E_b, a very long time delay will ensue and the system will resonate. As this transition probability increases, the resonant energy will increasingly depart from E_b, and usually the time delay will be reduced.

In summary, resonances will occur for elastic scattering when the system, restricted to other channels, as illustrated in Fig. 1.3, has bound states of positive energy, E_b. The resonance energies will be close[‡] to E_b. The coupling to the entrance channel need not be small. However, if it is strong, the shift of the resonance energy from E_b will be large and the width increased, that is, the time delay shortened.

Resonance scattering of light by atoms provides a well-known example. The incident projectile is a photon of energy $\hbar\omega$, the target an atom in its ground state. This system forms the entrance channel. This channel will couple to an excited state of the atom with excitation energy ε. In the *absence* of the electromagnetic coupling by virtue of which radiation back to the ground state would occur, this excited state is bound. When the photon energy $\hbar\omega$ is close to ε, $\hbar\omega \sim \varepsilon$, a resonance in the scattering of the photon will occur in which the incident photon is absorbed by the atom and then emitted.

This effect can also be observed in the passage of monochromatic light through a medium made up of such atoms. If again $\hbar\omega \sim \varepsilon$, the index of refraction, n, will undergo a very sharp change as a function of photon energy, as illustrated in Fig. 1.4, the phenomenon being referred to as *anomalous dispersion.*

Another example, and one that more closely approaches the nuclear case, considers the interactions of a projectile (e.g., a neutron) and a target nucleus in the ground state. The incident projectile moves in a field of force exerted by the nucleus, as illustrated in Fig. 1.5*a*. The target nucleus shown in the same

[‡]If E_b is near zero, the effect of coupling to the entrance channel can move the resonance energy to negative values. The resonance is then referred to as a negative energy resonance. This concept is useful if there is an effect of the resonance at positive energies.

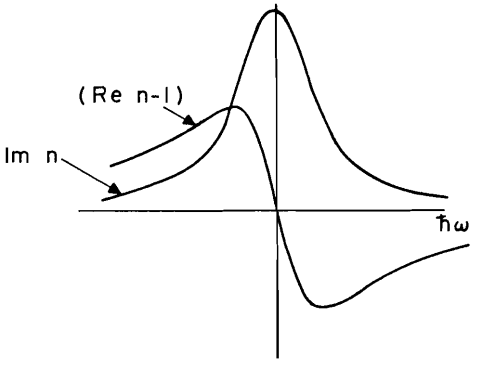

FIG. 1.4

figure consists, in the independent particle model, of nucleons, indicated by the filled circles, in the bound-state orbits. Figure 1.5*a* then illustrates the entrance channel. As a consequence of the interaction between the projectile and the target, a nucleon in the target is raised in energy while the incident projectile loses a corresponding amount of energy. If the excitation energy of the target is ε and the incident projectile energy is E, the energy of the projectile is $E - \varepsilon$. This situation is shown in Fig. 1.5*b*, where it is assumed that $E - \varepsilon$ is *negative*. The system is now bound in the sense that all the nucleons are individually bound. Of course, if the excitation energy of the target were to be returned to the projectile, it would again be unbound. But in the absense of that coupling the system is a bound one, although the *total* energy E is positive. The potential in which the projectile moves does have a bound state at $-E_b$, indicated by the line in the projectile half of the figure. According to the qualitative discussion presented earlier, when $E - \varepsilon$ is close to $-E_b$, or

$$E \sim \varepsilon - E_b$$

a resonance will occur. This example is still very far from a realistic description of the neutron–nucleus interaction, but it does contain the essential elements.

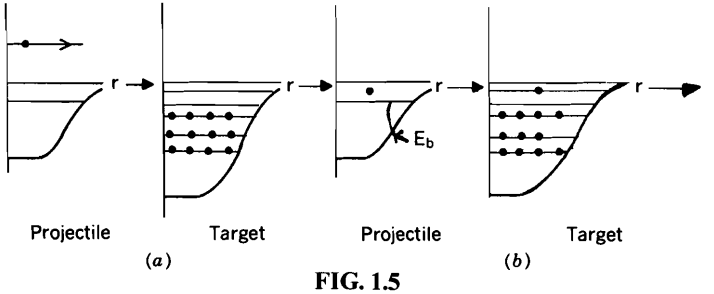

FIG. 1.5

A detailed, albeit simple example will serve to illustrate this discussion. The Hamiltonian of the system illustrated in Fig. 1.5 can be written

$$H = H_N + T + V(\mathbf{r}, \xi) \tag{1.1}$$

where $\mathbf{r}$ is the coordinate of the incident particle and ξ represents all the independent coordinates of the nucleons in the nucleus. T is the kinetic energy operator, $V(\mathbf{r}, \xi)$ the potential energy of the projectile relative to the target center of mass, and H_N the Hamiltonian for the target nucleus, therefore depending only on ξ. Assume for simplicity that this Hamiltonian has only two eigenvalues, 0 and ε, with the corresponding normalized wave functions $\psi_0(\xi)$ and $\psi_1(\xi)$. The total wave function for the system is then

$$\Psi = \frac{1}{r}[u_0(\mathbf{r})\psi_0(\xi) + u_1(\mathbf{r})\psi_1(\xi)] \tag{1.2}$$

where u_0 and u_1 describe the projectile wave functions in the two channels Fig. 1.5*a* and *b*, respectively. Inserting this expression for Ψ in the Schrödinger equation,

$$H\Psi = E\Psi$$

and using the orthogonality of ψ_0 and ψ_1 yields a pair of coupled equations for u_0 and u_1:

$$[E - (T + V_{00})]\frac{u_0}{r} = V_{01}\frac{u_1}{r}$$

$$[E - \varepsilon - (T + V_{11})]\frac{u_1}{r} = V_{10}\frac{u_0}{r}$$

where

$$V_{ij}(\mathbf{r}) = \langle \psi_i | V(\mathbf{r}, \xi) \psi_j \rangle_\xi$$

The integrations in these matrix elements is, as indicated, over the variables ξ only. We now assume that u_0 and u_1 are spherical, that V_{00} is given by an attractive square well,

$$V_{00} = -V_0 \qquad r \lesssim a$$

that

$$V_{11} = -\frac{\lambda\hbar^2}{2m}\delta(r - a) \qquad \lambda = \text{constant}$$

and that

$$V_{01} = V_{10} = -\frac{A\hbar^2}{2m}\delta(r-a) \qquad A = \text{constant}$$

Hence u_0 and u_1 satisfy the differential equations

$$u_0'' + (k^2 + K_0^2)u_0 = -A\delta(r-a)u_1 \qquad r \leqslant a \tag{1.3a}$$

$$u_1'' + (-\kappa^2 + \lambda\delta(r-a))u_1 = -A\delta(r-a)u_0 \tag{1.3b}$$

where

$$\kappa^2 = \frac{2m}{\hbar^2}(\varepsilon - E) \qquad K_0^2 = \frac{2m}{\hbar^2}V_0$$

and

$$k^2 = \frac{2m}{\hbar^2}E$$

This selection of potentials has the merit that (1.3*b*) in the absence of the coupling to the entrance channel has only one bound state at $\kappa = \kappa_0$; that is,

$$v_1'' + [-\kappa_0^2 + \lambda\delta(r-a)]v_1 = 0$$

has only one bound state solution. We leave it as an exercise for the reader to show that κ_0 satisfies the equation

$$\lambda a = \frac{2\kappa_0 a}{1 - e^{-2\kappa_0 a}} \tag{1.4}$$

The right-hand side of this equation is given by the solid curve in Fig. 1.6. When $\lambda a > 1$, one solution of (1.4) exists at the intersection of the curve and

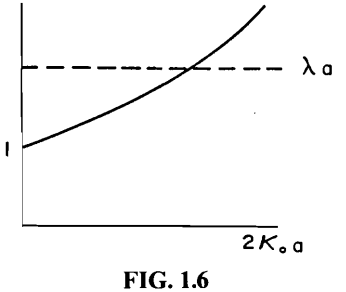

FIG. 1.6

the dashed line. As we shall see, the resonance generated by the coupled equations (1.3) will occur near the solution of (1.4)

As a first step we integrate (1.3*a*) and (1.3*b*) over a small range $a \pm \delta$. This yields

$$u_0'(a^+) - u_0'(a^-) = -Au_1(a)$$
$$u_1'(a^+) - u_1'(a^-) + \lambda u_1(a) = -Au_0(a)$$

Note that

$$u_0 = \sin Kr \qquad r \leqslant a$$
$$u_1 = \begin{cases} \alpha \sinh \kappa r & r \leqslant a \\ \alpha \sinh \kappa a e^{-\kappa(r-a)} & r \geqslant a \end{cases}$$

where

$$K = \sqrt{k^2 + K_0^2}$$

and α is a constant to be determined. It is now a simple matter to evaluate

$$f \equiv \frac{au_0'(a^+)}{u_0(a)}$$

One obtains

$$f = Ka \cot Ka + \frac{A^2a^2}{[2\kappa a/(1 - e^{-2\kappa a})] - \lambda a} \tag{1.5}$$

As we show in the Appendix to this chapter, resonances occur when $f = 0$. The second term in (1.5) is plotted in Fig. 1.7. Assuming that $k \ll K_0$, the first term is a constant that we take to be positive as an example. The function f equals zero at the intersection of the dashed line and the solid curve. The value of the resonance energy, E_R, is $E_R = \varepsilon - \hbar^2\kappa_R^2/2m$. This is a resonance if $E_R > 0$.

The difference between κ_R and κ_0 depends on the curvature of the second term as a function of κa and the size of the first. Assuming that $\kappa_R \sim \kappa_0$, one obtains

$$\kappa_R a \simeq \kappa_0 a + \frac{\kappa_0}{\lambda} \frac{A^2a^2}{(1 - \lambda a + 2\kappa_0 a)Ka \cot Ka}$$

This formula demonstrates that the deviation of the κ_R from κ_0, that is, the deviation of the resonance energy from the energy of the bound state of channel

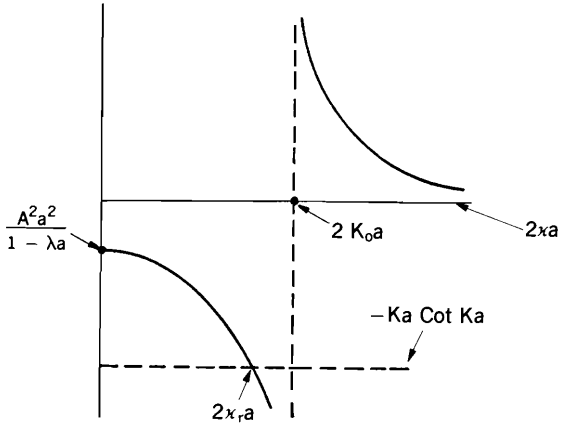

FIG. 1.7

u_1, grows as the strength of the coupling between the channels measured by Aa increases.

The width of the resonance is shown in the Appendix to this chapter to be given by

$$\Gamma = \frac{-2\kappa a}{(\partial f/\partial E)_{E=E_R}} \tag{1.6}$$

Evaluating the derivative at κ_0 rather than at κ_R, to simplify the result, and assuming Ka to be constant yields

$$\Gamma \simeq \frac{2\hbar^2 k\kappa_0^2}{m\lambda} \frac{A^2a^2}{[1+(2\kappa_0-\lambda)a](Ka\cot Ka)^2}$$

We see, as predicted, that the width increases (and the time delay decreases) as the coupling strength increases.

When E differs from E_R by an energy several times Γ, the energy variation of the cross section because of the energy variation of the second term on the right-hand side of (1.5) becomes unimportant. Of course, an effect remains, and usually the coupling to the second channel does increase the time delay.

In an actual projectile–nucleus reaction, the second channel is replaced by many channels labeled "other" channels in Fig. 1.1. There are then many bound states and correspondingly, many resonances, in contrast to the single bound state and single resonance of the coupled equations (1.3). In the next section the formalism applicable to the more complex and more realistic situation is developed.

2. FORMAL THEORY

The formal theory of nuclear reactions presented below relies on the concepts presented in the preceding section. Of course, the system to which it is applied, the compound system consisting of the projectile plus target nucleus, is much more complex than the system discussed at the end of Section 1 for which the target nucleus was assumed to have only two states, the ground state and one excited state. Nevertheless, it is possible by using projection operator techniques to rephrase the more general problem so that significant similarities to the simpler case are developed.

What we shall do is to partition all the states of the system into two sets of states. One set will contain the entrance channel and at least those channels that are involved in the prompt component of the reaction amplitude, that is, those which contribute to the direct single step as well as direct multistep processes for the reaction under consideration. This set of channels will play a role similar to that of $u_0\psi_0$ of (1.2). The second set will contain all the remaining channels (the other channels) including, necessarily, those that are bound in the absence of coupling to the first set. They play a role similar to that of $u_1\psi_1$ of (1.2).

These conditions do not constitute a precise definition of the partition. As we shall see, this lack of precision is useful since it allows the insertion of the pertinent physics of the problem into its analytical formulation. One illustration of such a partition would be helpful. Suppose that the physical process is the elastic and inelastic scattering of a pion by a nucleus (to avoid, in this illustration, the complications of the Pauli principle that would occur if the incident particle were a proton or, for that matter, any nuclear projectile). The appropriate set of states will include the entrance channel $u_0(\mathbf{r})\psi_0(\xi)$ as well as the inelastic states $u_n(\mathbf{r})\psi_n(\xi)$, where ψ_n are the states of the target nucleus, ψ_0 being the ground state, ψ_1 the first excited state, and so on; ξ are the coordinates, including spin and isospin of the nucleons making up the target. The functions $u_n(\mathbf{r})$ give the wave functions of the pion, where $\mathbf{r}$ is the pion coordinate relative to the center of mass of the target. If the energy of the projectile is E and the excitation energy corresponding to ψ_n is ε_n, the energy associated with the pion is $E-\varepsilon_n$. If this energy is positive, the asymptotic form of the wave function u_n, $n \neq 0$, will be of the form of an outgoing wave $e^{ik_n r}/r$, where $k_n = [2m/\hbar^2)(E-\varepsilon_n)]^{1/2}$. Those channels for which k_n is real, that is, $(E-\varepsilon_n) > 0$, are called *open channels*. If $E-\varepsilon_n$ is less than zero, k_n is pure imaginary and the asymptotic form of u_n is proportional to $e^{ik_n r}/r = e^{-|k_n|r}/r$. These channels are referred to as *closed channels*.

Once the representation of the Hilbert space to be used is described, the next task is partition into the prompt (the first set) and delaying (the second set) components. One solution is to include all the open channels in the first set and all the closed channels in the second set. This is the partition most commonly used at relatively low projectile energies, since then the number of open channels will be small. However, even in this case it sometimes proves advantageous to include some of the closed channels in the prompt space, or vice versa, some

of the open channels in the space of the delaying channels. This option is always available.

At high projectile energies, the delaying channels become unimportant for the most part. All important channels are prompt (although there can be delayed after-effects) and the partition used in Chapter II selects for the first set just those channels that are of immediate interest (e.g., the entrance channel if elastic scattering is being studied).

The selection of the set $u_n(\mathbf{r})\psi_n(\xi)$ is not just a matter of convenience. There is an implied statement in making that choice regarding the reaction mechanism. For example, it is implicitly asserted that particle transfer channels are not important for elastic or inelastic scattering. This is, in fact, not always the case, for it can occur that by transferring a particle or cluster of particles from the projectile to the target, it is possible at some energies easily to excite a "giant" resonance of the new system formed in this way. Inelastic excitation or elastic scattering of the target would follow if inverse particle transfer back to the projectile is made. In the case of pion–nucleus collision, the list given above of the possible channels in fact omitted one significant channel that is formed when the pion is absorbed by the target nucleon, the Δ.

These examples stress the flexibility of the partition and the role of physical intuition, first, in selecting the complete set that is to be partitioned and then, in choosing the partition.

Since the nature of the representation of the Hilbert space of the problem will vary considerably, it will be useful to develop a general formalism. Toward this end let us assume that the Hilbert space of the problem is partitioned as described above into two orthogonal components, $\mathscr{P}$ and $\mathscr{Q}$. The first of these, $\mathscr{P}$, will contain the prompt channels; the other, $\mathscr{Q}$, the closed channels, the exact nature of the partition depending on the physics of the reaction being considered, as described above. The projection operators P and Q project onto the subspace $\mathscr{P}$ and $\mathscr{Q}$, respectively, and satisfy

$$\begin{aligned} P &= P^\dagger \qquad Q = Q^\dagger \\ P^2 &= P \qquad Q^2 = Q \\ P + Q &= 1 \end{aligned} \tag{2.1}$$

The state vector of the system, Ψ, satisfies the Schrödinger equation

$$(E - H)\Psi = 0 \tag{2.2}$$

We can now determine the equation satisfied by the prompt component of Ψ, $P\Psi$, and the time-delaying component, $Q\Psi$. Writing

$$\Psi = P\Psi + Q\Psi$$

and multiplying (2.2) from the left by P and Q yields

$$(E - H_{PP})(P\Psi) = H_{PQ}(Q\Psi) \tag{2.3}$$

and

$$(E - H_{QQ})(Q\Psi) = H_{QP}(P\Psi) \tag{2.4}$$

where

$$H_{PP} \equiv PHP, \quad H_{PQ} \equiv PHQ, \quad \text{etc.}$$

If we write

$$H = H_0 + V$$

where H_0 is the sum of the Hamiltonians for the internal degrees of freedom of the projectile and target and the kinetic energy operator for their relative motion. It is usual to restrict the class of projection operators to those for which

$$H_{PQ} = V_{PQ} \qquad H_{QP} = V_{QP}$$

The analogy of (2.3) and (2.4) with (1.3) of Section 1 is rather clear. Whereas in the latter case $\mathscr{P}$ and $\mathscr{Q}$ contain only one component $(1/r)u_0\psi_0$ and $(1/r)u_1\psi_1$, respectively, the more general case of (2.3) and (2.4) will contain many components.

This simple partition can be used to obtain some quite general results. We can formally solve (2.4) as follows:

$$Q\Psi = \frac{1}{E^{(+)} - H_{QQ}} H_{QP}P\Psi \tag{2.5}$$

This expression includes the boundary condition that there is no incident wave in the subspace $\mathscr{Q}$. The $i\eta$ in $E^{(+)} = E + i\eta$, $\eta \to 0^+$, is included in case some of the open channels are in $\mathscr{Q}$. Substituting (2.5) in (2.3) yields

$$(E - H_{\text{eff}})P\Psi = 0 \tag{2.6}$$

where

$$H_{\text{eff}} = H_{PP} + H_{PQ}\frac{1}{E^{(+)} - H_{QQ}} H_{QP} \tag{2.7}$$

The first term on the right-hand side, H_{PP}, is associated with the prompt process. The second term describes the time-delaying effect of coupling $\mathscr{P}$ space with $\mathscr{Q}$, propagation in $\mathscr{Q}$ as given by $1/(E^{(+)} - H_{QQ})$ and then reemission into $\mathscr{P}$. The Schrödinger equation (2.6) iterates this process.

Problem. Referring to the discussion in deShalit and Feshbach (74, p. 648), show that $\mathscr{H}_{\text{eff}}$, where

$$H_{\text{eff}} = P\mathscr{H}_{\text{eff}}P$$

satisfies the integral equation

$$\mathscr{H}_{\text{eff}} = H + HQ\frac{1}{E^{(+)} - H_0}Q\mathscr{H}_{\text{eff}} \tag{2.8}$$

We see that H_{eff} is energy dependent, complex, and nonlocal. These properties are consequences of the presence of the propagator $1/(E^{+} - H_{QQ})$. Because of this, there is a dispersion-type relationship between the real and imaginary parts of $\mathscr{H}_{\text{eff}}$. To derive this result most simply, we express $H_{\text{eff}} - H_{PP}$ in terms of the eigenstates of the operator H_{QQ}. This operator will generally have bound eigenstates whose eigenvalues form a discrete spectrum, and unbound eigenstates whose eigenvalues form a continuous spectrum. Hence

$$\begin{aligned} H_{QQ}\Phi_s &= \mathscr{E}_s\Phi_s \qquad \text{(discrete)} \\ H_{QQ}\Phi(\mathscr{E},\alpha) &= \mathscr{E}\Phi(\mathscr{E},\alpha) \end{aligned} \tag{2.9}$$

where α is an index which together with $\mathscr{E}$ specifies the continuum state. These state vectors are assumed to be normalized and therefore form an orthonormal set:

$$\begin{aligned} &\langle \Phi_s | \Phi_{s'} \rangle = \delta_{ss'} \qquad \langle \Phi_s | \Phi(\mathscr{E},\alpha) \rangle = 0 \\ &\langle \Phi(\mathscr{E}',\alpha') | \Phi(\mathscr{E},\alpha) \rangle = \delta(\mathscr{E}' - \mathscr{E})\delta(\alpha - \alpha') \end{aligned}$$

Then

$$H_{\text{eff}} - H_{PP} = \sum_s \frac{H_{PQ}\Phi_s\rangle\langle\Phi_s H_{QP}}{E - \mathscr{E}_s} + \int d\alpha \int d\mathscr{E} \frac{H_{PQ}\Phi(\mathscr{E},\alpha)\rangle\langle\Phi(\mathscr{E},\alpha)H_{QP}}{E^{(+)} - \mathscr{E}} \tag{2.10}$$

Using the result

$$\frac{1}{E^{(+)} - \mathscr{E}} = \frac{\mathscr{P}}{E - \mathscr{E}} - i\pi\delta(E - \mathscr{E})$$

where $\mathscr{P}$ indicates that the principal value integration should be taken and $\delta(E - \mathscr{E})$ is the Dirac delta function, we obtain

$$\text{Re}(H_{\text{eff}} - H_{PP}) = \sum_s \frac{H_{PQ}\Phi_s\rangle\langle\Phi_s H_{QP}}{E - \mathscr{E}_s} + \mathscr{P}\int \frac{d\mathscr{E}}{E - \mathscr{E}} \int d\alpha H_{PQ}\Phi(\mathscr{E},\alpha)\rangle\langle\Phi(\mathscr{E},\alpha)H_{QP} \tag{2.11a}$$

while

$$\text{Im}(H_{\text{eff}} - H_{PP}) = -\pi \int d\alpha H_{PQ}\Phi(\mathscr{E},\alpha)\rangle\langle\Phi(\mathscr{E},\alpha)H_{QP} \tag{2.11b}$$

Note that this operator is negative definite, as it must be to represent a decrease in flux in the prompt channels because of the presence of open channels in $\mathcal{Q}$ space. Substituting (2.11*b*) into (2.11*a*) yields

$$\operatorname{Re} H_{\text{eff}} = H_{PP} + \sum_s \frac{H_{PQ}\Phi_s\rangle\langle\Phi_s H_{QP}}{E - \mathscr{E}_s} - \frac{1}{\pi}\mathscr{P}\int \frac{d\mathscr{E}}{E - \mathscr{E}} \operatorname{Im} H_{\text{eff}}(\mathscr{E}) \tag{2.12}$$

where it is assumed that H_{PP} is real, as is usually the case.

Problem. Show that (2.12) can also be written

$$\operatorname{Re} V_{\text{eff}} = V_{PP} + \sum_s \frac{V_{PQ}\Phi_s\rangle\langle\Phi_s V_{QP}}{E - \mathscr{E}_s} - \frac{1}{\pi}\mathscr{P}\int \frac{d\mathscr{E}}{E - \mathscr{E}} \operatorname{Im} V_{\text{eff}}(\mathscr{E}) \tag{2.13}$$

This result is the nuclear analog of the Kronig–Kramers relation between the real and imaginary parts of the index of refraction for light (see Fig. 1.4) propagating through an infinite medium. For an infinite nuclear medium there are no significant closed channels at low excitation energies, so that the second term on the right-hand side of (2.13) is essentially zero. However, at energies that permit the excitation of isobars, this term could be appreciable.

We turn next to resonances. Recall from the simple example of the preceding section [see (1.3) et seq.] that resonances are associated with the bound states of the time-delaying component of the wave function. In the present context, this is $Q\Psi$. The bound states are now given by Φ_s, with energies $\mathscr{E}_s$. As one can see from (2.10) for H_{eff}, rapid energy dependence of H_{eff} will occur near $\mathscr{E}_s$, and one would predict that rapid energy dependence and therefore a long time delay will occur for $E \simeq \mathscr{E}_s$. For E near $\mathscr{E}_s$, H_{eff} can be written

$$H_{\text{eff}} = \bar{H}_{PP} + \frac{H_{PQ}\Phi_s\rangle\langle\Phi_s H_{QP}}{E - \mathscr{E}_s} \tag{2.14}$$

where all the remaining terms in the expansion (2.10) have been grouped together to form $\bar{H}_{PP}$, which will be assumed to have a slow energy dependence near $\mathscr{E}_s$. We shall see what this means after examining its consequences. Equation (2.14) is equivalent to the partition in which $\mathcal{Q}$ contains only one state Φ_s.

The Schrödinger equation (2.6) becomes

$$(E - \bar{H}_{PP})(P\Psi) = \frac{H_{PQ}\Phi_s\rangle\langle\Phi_s|H_{QP}P\Psi\rangle}{E - \mathscr{E}_s}$$

Its formal solution is given by

$$P\Psi = \chi_i^{(+)} + \frac{1}{E^{(+)} - \bar{H}_{PP}} \frac{H_{PQ}\Phi_s\rangle\langle\Phi_s|H_{QP}P\Psi\rangle}{E - \mathscr{E}_s} \tag{2.15}$$

where $\chi_i^{(+)}$, the distorted incident wave, is the solution of

$$(E - \bar{H}_{PP})\chi_i^{(+)} = 0$$

satisfying the outgoing wave boundary condition. The $\mathcal{T}$ matrix giving the transition amplitude is

$$\mathcal{T}_{fi} = \mathcal{T}_{fi}^{(P)} + \frac{\langle \chi_f^{(-)} | H_{PQ} \Phi_s \rangle \langle \Phi_s | H_{QP} P\Psi \rangle}{E - \mathscr{E}_s} \tag{2.16}$$

where $\mathcal{T}_{fi}^{(P)}$ is the prompt direct amplitude associated with the first term of (2.15), and $\chi_f^{(-)}$ is the solution

$$(E - \bar{H}_{PP})\chi_f^{(-)*} = 0$$

describing the final amplitude satisfying incoming wave boundary conditions. Returning to (2.15), multiply both sides by $\langle \Phi_s H_{QP}$, yielding

$$\langle \Phi_s | H_{QP} P\Psi \rangle = \langle \Phi_s | H_{QP} \chi_i^{(+)} \rangle + \frac{\langle \Phi_s | W_{QQ} \Phi_s \rangle \langle \Phi_s | H_{QP} P\Psi \rangle}{E - \mathscr{E}_s}$$

where

$$W_{QQ} \equiv H_{QP} \frac{1}{E^{(+)} - \bar{H}_{PP}} H_{PQ} \tag{2.17}$$

Solving for $\langle \Phi_s | H_{QP} P\Psi \rangle$, one obtains

$$\langle \Phi_s | H_{QP} P\Psi \rangle = \frac{(E - \mathscr{E}_s)\langle \Phi_s | H_{QP} \chi_i^{(+)} \rangle}{E - \mathscr{E}_s - \langle \Phi_s | W_{QQ} \Phi_s \rangle}$$

Inserting this result in (2.16) yields

$$\mathcal{T}_{fi} = \mathcal{T}_{fi}^{(P)} + \frac{\langle \chi_f^{(-)} | H_{PQ} \Phi_s \rangle \langle \Phi_s | H_{QP} \chi_i^{(+)} \rangle}{E - \mathscr{E}_s - \langle \Phi_s | W_{QQ} \Phi_s \rangle} \tag{2.18}$$

This is a typical resonance formula. As expected, the resonance occurs near $\mathscr{E}_s$. There is an energy shift and a width that are given by the expectation value of W_{QQ}:

$$\langle \Phi_s | W_{QQ} \Phi_s \rangle = \Delta_s(E) - \tfrac{1}{2} i \Gamma_s(E) \tag{2.19}$$

so that

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \frac{\langle \chi_f^{(-)} | H_{PQ} \Phi_s \rangle \langle \Phi_s | H_{QP} \chi_i^{(+)} \rangle}{E - E_s + (i/2)\Gamma_s} \tag{2.20}$$

Note that E_s and Γ_s are functions of E. The statement that E_s is the resonance energy assumes that their energy dependence is weak. Inserting W_{QQ} from (2.17) into (2.19) provides more explicit expressions for Δ_s and Γ_s:

$$\begin{aligned}\langle \Phi_s | W_{QQ} \Phi_s \rangle &= \left\langle \Phi_s \middle| H_{QP} \frac{1}{E^{(+)} - \bar{H}_{PP}} H_{PQ} \Phi_s \right\rangle \\ &= \left\langle \Phi_s \middle| H_{QP} \frac{\mathscr{P}}{E - \bar{H}_{PP}} H_{PQ} \Phi_s \right\rangle - i\pi \langle \Phi_s | H_{QP} \delta(E - \bar{H}_{PP}) H_{PQ} \Phi_s \rangle\end{aligned}$$

Comparing with (2.19), we see that

$$\Gamma_s = 2\pi \langle \Phi_s | H_{QP} \delta(E - \bar{H}_{PP}) H_{PQ} \Phi_s \rangle \tag{2.21}$$

Introducing a complete set of states $\chi_r^{(+)}$ of the Hamiltonian $\bar{H}_{PP}$ at the energy E, one obtains

$$\Gamma_s = 2\pi \sum_r |\langle \Phi_s | H_{QP} \chi_r^{(+)} \rangle|^2 \equiv \sum \Gamma_{sr}(E) \tag{2.22}$$

The width Γ_s is thus the sum of *partial widths* Γ_{sr} corresponding to the decay of the state Φ_s into various possible channels with the same value of the energy E. The numerator of the resonance term in (2.20) is obviously related to these partial widths:

$$\begin{aligned}\langle \Phi_s | H_{QP} \chi_i^{(+)} \rangle &= e^{i\delta_i} \sqrt{\frac{\Gamma_{si}}{2\pi}} \equiv e^{i\delta_i} \frac{g_{si}}{\sqrt{2\pi}} \\ \langle \chi_f^{(-)} | H_{PQ} \Phi_s \rangle &= e^{i\delta_f} \sqrt{\frac{\Gamma_{sf}}{2\pi}} \equiv \frac{e^{i\delta_f} g_{sf}}{\sqrt{2\pi}}\end{aligned} \tag{2.23}$$

where δ_i gives the phase of $\chi_i^{(+)}$, that is,

$$\chi_i^{(+)} = e^{i\delta_i} F_i \tag{2.24}$$

where F_i is real. Since Φ_s is a bound-state wave function, it can be taken as real. These results, (2.22) and (2.23), upon insertion into (2.20), yield the familiar

Breit–Wigner formula‡

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \frac{1}{2\pi} e^{i(\delta_i + \delta_f)} \frac{g_{sf} g_{si}}{E - E_s + \frac{1}{2} i\Gamma_s} \tag{2.25}$$

Further properties of Γ_{sr} will be discussed later (see p. 242).

The subscript s represents all the quantum numbers required to specify the bound state Φ_s. These will certainly include its total angular momentum J and its parity Π. The resonance therefore contributes to a particular partial wave, while the prompt, direct amplitude $\mathscr{T}_{fi}^{(P)}$ will generally involve several partial waves.

Problem. Show that

$$\Delta_s = \frac{1}{2\pi} \mathscr{P} \int \frac{dE' \Gamma_s(E')}{E - E'}$$

Show that

$$\sum_s \Gamma_s = 2\pi \sum_r \langle \chi_r^{(+)} | (H - \bar{H}_{PP})^2 | \chi_r^{(+)} \rangle$$

This one resonance + direct amplitude description is exact. It is not, however, useful if the direct amplitude $\mathscr{T}_{fi}^{(P)}$ and therefore $E_s(E)$ and $\Gamma_s(E)$ vary rapidly with energy E near E_s. Generally, if the resonances arising from the bound states at $\mathscr{E}_s$ are well separated, the direct amplitude is observed to vary slowly with energy. However, if the separation in energy D between neighboring $\mathscr{E}_s$ is on the order or less than Γ_s, it would be necessary to remove a group of resonances in, say, an energy interval d, before the direct amplitude would become relatively constant over the energy region in the center of d. The resonances are then said to *overlap*. The formalism developed above can readily

‡Equation (2.24) assumes that $\chi_i^{(+)}$ and $\chi_f^{(-)}$ are eigenstates of S_p, the S matrix associated with $\bar{H}_{PP}$, with eigenvalues $e^{2i\delta_i}$ and $e^{2i\delta_f}$, respectively. However, if they are solutions of coupled equations, that is, if the prompt space contains more than one channel, as will occur if direct inelastic or transfer reactions are energetically allowed, that will no longer be the case. It is then necessary to introduce the real orthogonal transformation $M_{i\alpha}$, which connects an eigenfunction of S_p, χ_α with eigenvalues $e^{2i\Delta_\alpha}$, with an open-channel distorted plane wave such as $\chi_i^{(+)}$. This transformation is obtained in the course of solving the prompt problem. Then (2.25) is replaced by

$$\mathscr{T}_{fi} = \sum M_{\alpha f} \left[\frac{1}{2\pi i} (1 - e^{2i\Delta_\alpha}) \delta_{\alpha\beta} + \frac{e^{i(\Delta_\alpha + \Delta_\beta)}}{2\pi} \frac{g_{s\alpha} g_{s\beta}}{E - E_s + \frac{1}{2} i\Gamma_s} \right] M_{\beta i} \tag{2.26}$$

where $g_{s\alpha}$ gives the magnitude of $\langle \Phi_s | H_{QP} \chi_\alpha \rangle$. Results (2.26) holds in the DWA approximation, or when only one channel is open. We shall refer to the latter as the single-channel case.

be extended to include this case [Feshbach (58,62)]. However, it will be convenient and instructive to use a different but equivalent approach.

We begin with the general expression that follows from (2.3) and the two-potential theorem:

$$\mathcal{T}_{fi} = \mathcal{T}_{fi}^{(P)} + \langle \chi_f^{(-)} | H_{PQ} Q \Psi_i \rangle \tag{2.27}$$

where we have inserted the subscript i on Ψ to indicate that it is the Ψ developed by the incident wave $\chi_i^{(+)}$. The exact expression for $Q\Psi_i$ can be obtained as follows: Solve (2.3) for $P\Psi_i$:

$$P\Psi_i = \chi_i^{(+)} + \frac{1}{E^{(+)} - H_{PP}} H_{PQ} Q \Psi_i \tag{2.27'}$$

Substituting this result into (2.4) yields the inhomogeneous equation

$$(E - H_{QQ} - W_{QQ}) Q\Psi_i = H_{QP} \chi_i^{(+)} \tag{2.28}$$

where W_{QQ} is now given by

$$W_{QQ} \equiv H_{QP} \frac{1}{E^{(+)} - H_{PP}} H_{PQ} \tag{2.29}$$

Solving (2.28), one obtains

$$Q\Psi_i = \frac{1}{E - H_{QQ} - W_{QQ}} H_{QP} \chi_i^{(+)} \tag{2.30}$$

so that $\mathcal{T}_{fi}$ becomes

$$\mathcal{T}_{fi} = \mathcal{T}_{fi}^{(P)} + \left\langle \chi_f^{(-)} \middle| H_{PQ} \frac{1}{E - H_{QQ} - W_{QQ}} H_{QP} \chi_i^{(+)} \right\rangle \tag{2.30'}$$

The second term gives explicitly the time-delayed component that is generated by the coupling of the $\mathcal{P}$ space to the $\mathcal{Q}$ space. Its explicit energy dependence, as given by the propagator $[1/(E - H_{QQ} - W_{QQ})]$, can be rapid when E is near a pole of the propagator.

One can recover the result of (2.18) if $\mathcal{Q}$ space contains only one state Φ_s satisfying (2.9). Then

$$\begin{aligned} \frac{1}{E - H_{QQ} - W_{QQ}} &= |\Phi_s\rangle \left\langle \Phi_s \middle| \frac{1}{E - H_{QQ} - W_{QQ}} \Phi_s \right\rangle \langle \Phi_s| \\ &= |\Phi_s\rangle \frac{1}{E - \langle \Phi_s | (H_{QQ} + W_{QQ}) \Phi_s \rangle} \langle \Phi_s = \Phi_s | \rangle \frac{1}{E - \mathscr{E}_s - \Delta_s + \frac{1}{2} i \Gamma_s} \langle \Phi_s| \end{aligned}$$

where (2.19) has been used. Upon inserting this last result into (2.30′). one again obtains the Breit–Wigner formula, (2.20).

The extension of this procedure to the case of several overlapping resonances is straightforward. We simply expand $\mathcal{Q}$ space to include just the requisite number of bound states Φ_s. That number is determined by the requirement that the energy dependence of the amplitude generated by H_{PP} in the energy interval of interest is sufficiently slow. One can then obtain the eigenvalues of the operator $H_{QQ} + W_{QQ}$ by solving the secular equation

$$\det|(E_\mu - \mathscr{E}_s)\delta_{st} - W_{st}| = 0 \tag{2.31}$$

where

$$W_{st} \equiv \langle \Phi_s | W_{QQ} \Phi_t \rangle = \left\langle \Phi_s \middle| H_{QP} \frac{1}{E^{(+)} - H_{PP}} H_{PQ} \Phi_t \right\rangle \tag{2.32}$$

The eigenvalues E_μ are complex, since W_{st} is not Hermitian. The result for the isolated resonance is obtained immediately if the determinant has only one term. The eigenfunctions of $H_{QQ} + W_{QQ}$, Ω_μ, are finite linear combinations of the bound-state wave functions Φ_s which have been included in $\mathcal{Q}$:

$$\Omega_\mu = \sum x_s^{(\mu)} \Phi_s$$

The functions Ω_μ form a biorthogonal set with the adjoint functions $\Omega_\mu^{(A)}$:

$$\langle \Omega_\mu^{(A)} | \Omega_\nu \rangle = \delta_{\mu\nu}$$

The coefficients $x_s^{(\mu)}$ satisfy the set of linear equations

$$\sum_t [(E_\mu - \mathscr{E}_s)\delta_{st} - W_{st}] x_t^{(\mu)} = 0$$

while the corresponding coefficient for $\Omega_\nu^{(A)}$, $\tilde{x}_s^{(\nu)}$ satisfies

$$\sum_s \tilde{x}_s^{(\mu)} [(E_\mu - \mathscr{E}_s)\delta_{st} - W_{st}] = 0$$

With these results in hand it is now a simple matter to expand $1/(E - H_{QQ} - W_{QQ})$ in terms of Ω_μ and so finally obtain

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \sum_\mu \frac{A_{fi}^{(\mu)}}{E - E_\mu} \tag{2.33}$$

where

$$A_{fi}^{(\mu)} = \langle \chi_f^{(-)} | H_{PQ} \Omega_\mu \rangle \langle \Omega_\mu^{(A)} | H_{QP} \chi_i^{(+)} \rangle \tag{2.34}$$

Note that the $A_{fi}^{(\mu)}$ are complex and that

$$\sum_{\mu} A_{fi}^{(\mu)} = \sum_{r} \langle \chi_f^{(-)} | H_{PQ} \Phi_s \rangle \langle \Phi_s | H_{QP} \chi_i^{(+)} \rangle \tag{2.35}$$

where the completeness of Φ_s has been used.

Let us for simplicity again consider a single channel, so that

$$\chi_f^{(-)} = e^{-2i\delta} \chi_i^{(+)}$$

Then

$$\sum_{\mu} A_{fi}^{(\mu)} = e^{2i\delta} \sum_{s} |\langle \chi_i^{(+)} | H_{PQ} \Phi_s \rangle|^2 \tag{2.36}$$

The $A^{(\mu)}$ and E_μ are not independent. For example, the diagonal sum rule when applied to the secular equation (2.31) yields

$$\sum_{\mu} \operatorname{Im} E_\mu = \sum_{s} \operatorname{Im} W_{ss}$$

Inserting the value of W_{ss} and making use of (2.36), one obtains

$$\sum_{\mu} A_{fi}^{(\mu)} = -\frac{1}{\pi} e^{2i\delta} \sum_{\mu} \operatorname{Im} E_\mu \tag{2.37}$$

For isolated resonances this equality holds for each individual μ, thus reducing to (2.22), which states that the total width ($\equiv -2 \operatorname{Im} E^{(\mu)}$) equals the sum of the partial widths that appear in the numerator of the resonance amplitude. Equation (2.37) states that this equality holds on the average, that is,

$$2\pi \sum_{s} |\langle \chi_i^{(+)} | H_{PQ} \Phi_s \rangle|^2 = -2 \sum_{\mu} \operatorname{Im} E_\mu \tag{2.38}$$

Other relationships, such as (2.37), may be obtained from the properties of the secular equation (2.31).

Because of the relationships between $A^{(\mu)}$ and E_μ, the number of independent real numbers required to describe the resonance term in (2.33) is not as large as might appear. Without these connections each term involves two complex numbers, $A_{fi}^{(\mu)}$ and $E^{(\mu)}$, or four real numbers, We shall shortly derive a more economical expression in which this number of real parameters is reduced by one-half.

It is instructive at this point to show that relationship (2.37) is a consequence of the unitarity of the S matrix. For this purpose it will suffice to develop this relation for single-channel elastic scattering. Then

$$\mathscr{T} = \mathscr{T}_p + \sum_{\mu} \frac{A^{(\mu)}}{E - E_\mu}$$

From

$$S = 1 - 2\pi i\mathscr{T}$$

one obtains

$$S = S_p - 2\pi i \sum \frac{A^{(\mu)}}{E - E_\mu}$$

where S_p is the S matrix for the prompt process. It is convenient to introduce $\mathscr{A}^{(\mu)}$, which is defined by

$$A^{(\mu)} = S_p \mathscr{A}^{(\mu)}$$

Hence

$$S = S_p \left[1 - 2\pi i \sum \frac{\mathscr{A}^{(\mu)}}{E - E_\mu} \right]$$

To avoid obfuscating complications, consider the simple case of just two resonances,

$$\begin{aligned} S &= S_p \left[1 - 2\pi i \left(\frac{\mathscr{A}_1}{E - E_1} + \frac{\mathscr{A}_2}{E - E_2} \right) \right] \\ &= S_p \left[\frac{(E - E_1)(E - E_2) - 2\pi i \mathscr{A}_1 (E - E_2) - 2\pi i \mathscr{A}_2 (E - E_1)}{(E - E_1)(E - E_2)} \right] \end{aligned}$$

Unitarity, $SS^\dagger = 1$, requires that

$$(E - E_1)(E - E_2) - 2\pi i \mathscr{A}_1 (E - E_2) - 2\pi i \mathscr{A}_2 (E - E_2) = e^{i\lambda (E - E_1^*)(E - E_2^*)}$$

where the phase λ is to be determined. Equating coefficients of E^2 on both sides of the last equation gives $\lambda = 0$, while equating the coefficients of E yields

$$E_1 + E_2 + 2\pi i (\mathscr{A}_1 + \mathscr{A}_2) = E_1^* + E_2^*$$

so that

$$\mathscr{A}_1 + \mathscr{A}_2 = \frac{1}{\pi} (\mathrm{Im}\, E_1 + \mathrm{Im}\, E_2)$$

or

$$A_1 + A_2 = -\frac{1}{\pi} S_p (\mathrm{Im}\, E_1 + \mathrm{Im}\, E_2)$$

in agreement with (2.37).

Problem. Consider the case where several resonances contribute.

It is clear that many other relationships can be obtained by comparing coefficients of still lower powers of E. Not all of these relationships are

independent. It thus becomes important to determine the number of independent constants appearing (2.33).

The $\mathscr{K}$ Reactance Matrix (Feshbach (67a)). Consider (2.28) once more, where, it is to be recalled, $\mathscr{Q}$ space contains a finite number of bound states Φ_s. Expressing W_{QQ} more explicitly, (2.28) becomes

$$\left(E - H_{QQ} - H_{QP}\frac{\mathscr{P}}{E - H_{PP}}H_{PQ}\right)Q\Psi + i\pi\sum_{\gamma} H_{QP}\chi_{\gamma}^{(+)}\rangle\langle\chi_{\gamma}^{(+)}|H_{PQ}Q\Psi\rangle = H_{QP}\chi_i^{(+)} \tag{2.39}$$

where γ designates those open channels whose energy equals the initial energy, E. The wave functions for the incident state $\chi_i^{(+)}$ belong to the set $\chi_{\gamma}^{(+)}$, while $\chi_f^{(-)}$ belongs to a corresponding set of states $\chi_{\gamma}^{(-)}$. To simplify the formal manipulations it is useful to introduce the projection operator O:

$$O \equiv \sum_{\gamma} \chi_{\gamma}^{(+)}\rangle\langle\chi_{\gamma}^{(+)} = \sum_{\gamma} \chi_{\gamma}^{(-)}\rangle\langle\chi_{\gamma}^{(-)} \tag{2.40}$$

with

$$O^2 = 0 \qquad OQ = 0$$

Also, let

$$H_{QQ} + H_{QP}\frac{\mathscr{P}}{E - H_{PP}}H_{PQ} \equiv \mathscr{H}_{QQ}$$

so that (2.39) becomes

$$(E - \mathscr{H}_{QQ})Q\Psi + i\pi H_{QO}H_{OQ}Q\Psi = H_{QO}\chi_i^{(+)} \tag{2.41}$$

where

$$H_{OQ} \equiv OHQ$$

From (2.41)

$$Q\Psi = \frac{1}{E - \mathscr{H}_{QQ} + i\pi H_{QO}H_{OQ}}H_{QO}\chi_i^{(+)}$$

Note that

$$\begin{aligned}\frac{1}{E - \mathscr{H}_{QQ} + i\pi H_{QO}H_{OQ}}H_{QO} &= \frac{1}{1 + i\pi[1/(E - \mathscr{H}_{QQ})]H_{QO}H_{OQ}}\frac{1}{E - \mathscr{H}_{QQ}}H_{QO} \\ &= \frac{1}{E - \mathscr{H}_{QQ}}H_{QO}\frac{1}{1 + i\pi H_{OQ}[1/(E - \mathscr{H}_{QQ})]H_{QO}}\end{aligned}$$

where we have made use of the operator identities

$$\frac{1}{1+AB}A = A\frac{1}{1+BA} \qquad \frac{1}{XY} = \frac{1}{Y}\frac{1}{X}$$

Inserting the resulting expression for $Q\Psi$ into (2.27) yields

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \left\langle \chi_f^{(-)} \middle| \frac{\mathscr{K}}{1+i\pi\mathscr{K}} \chi_i^{(+)} \right\rangle \tag{2.42}$$

where

$$\mathscr{K} \equiv H_{oQ} \frac{1}{E - \mathscr{H}_{QQ}} H_{Qo} \tag{2.43}$$

Note that $\mathscr{K}$ is a finite Hermitian matrix for real E. We also note that the S matrix

$$S = 1 - 2\pi i\mathscr{T} = S_P - \frac{2\pi i\mathscr{K}}{1+i\pi\mathscr{K}}$$

or

$$S_{fi} = \langle \chi_f^{(-)} | \chi_i^{(+)} \rangle - \left\langle \chi_f^{(-)} \middle| \frac{2\pi i\mathscr{K}}{1+i\pi\mathscr{K}} \chi_i^{(+)} \right\rangle$$

Finally,

$$S_{fi} = \left\langle \chi_f^{(-)} \middle| \frac{1-i\pi\mathscr{K}}{1+i\pi\mathscr{K}} \chi_i^{(+)} \right\rangle \tag{2.44}$$

The unitarity of the S matrix can be demonstrated from this result. Note, however, that it is not immediate, but requires the use of the relation between the distorted waves $\chi_i^{(+)}$, $\chi_f^{(-)}$ and the undistorted waves.

Problem. Prove the unitarity of S_{fi} of (2.44).

To obtain a resonance expansion of $\mathscr{T}$, we introduce the eigenvalues κ_μ and eigenfunctions of the operator $\mathscr{K}$, which is Hermitian when E is real:

$$\mathscr{K} Y_\mu = \kappa_\mu Y_\mu \tag{2.45}$$

The functions Y_μ form an orthonormal set. They are also eigenfunctions of $\mathscr{S}$ where

$$\mathscr{S} = \frac{1-i\pi\mathscr{K}}{1+i\pi\mathscr{K}}$$

with eigenvalues $(1 - i\pi\kappa_\mu)/(1 + i\pi\kappa_\mu)$. If we write the eigenvalues of $\mathscr{S}$ as $e^{2i\sigma_\mu}$, we find that

$$\tan \sigma_\mu = -\pi\kappa_\mu$$

It is possible to express κ_μ in terms of e_s, the eigenvalues of $\mathscr{H}_{QQ}$, with corresponding eigenstates Φ_s. We note that

$$\kappa_\mu = \langle Y_\mu | \mathscr{K} Y_\mu \rangle = \left\langle Y_\mu \middle| H_{OQ} \frac{1}{E - \mathscr{H}_{QQ}} H_{QO} Y_\mu \right\rangle$$

or

$$\kappa_\mu = \sum_s \frac{|\langle Y_\mu | H_{OQ} \Phi_s \rangle|^2}{E - e_s}$$

Defining the widths

$$\gamma_{\mu s}^2 \equiv 2\pi |\langle Y_\mu | H_{OQ} \Phi_s \rangle|^2$$

we obtain finally‡

$$\kappa_\mu = \frac{1}{2\pi} \sum \frac{\gamma_{\mu s}^2}{E - e_s} \tag{2.46}$$

Note that $\gamma_{\mu s}$ and e_s are energy independent if the projection operator Q is chosen to be fixed with energy.

The transition matrix is then given by

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \sum_\mu \langle \chi_f^{(-)} | Y_\mu \rangle \frac{\kappa_\mu}{1 + i\pi\kappa_\mu} \langle Y_\mu | \chi_i^{(+)} \rangle \tag{2.47}$$

while

$$S_{fi} = \sum_\mu \langle \chi_f^{(-)} | Y_\mu \rangle \frac{1 - i\pi\kappa_\mu}{1 + i\pi\kappa_\mu} \langle Y_\mu | \chi_i^{(+)} \rangle = \sum_\mu \langle \chi_f^{(-)} | Y_\mu \rangle e^{2i\sigma_\mu} \langle Y_\mu | \chi_i^{(+)} \rangle \tag{2.48'}$$

It is often convenient to work in a representation in which S is diagonal and project to the scattering states $\chi^{(\pm)}$, as given explicitly by (2.48'). In the single-channel case the potential S matrix, $\langle \chi_f^{(-)} | \chi_i^{(+)} \rangle$, and the full S matrix are diagonal simultaneously. The resonance energies are given by the poles of (2.48'), that is, when

$$1 + i\pi\kappa_\mu = 0$$

‡Note that κ_μ is real for real E, with poles on the real axis with positive residues. However, κ_μ is not rigorously an R function in Wigner's terminology because e_s and Φ_s are functions of the energy.

or

$$1+\frac{i}{2}\sum\frac{\gamma_{\mu s}^2}{E-e_s}=0 \tag{2.48''}$$

For an isolated resonance, that is, e_s takes on only one value e_1, this equation yields the resonance energy, $e_1 - i\gamma_{\mu 1}/2$. The shift Δ_s no longer appears because of the difference between H_{QQ} and $\mathscr{H}_{QQ}$. To compare the result directly with (2.23), consider the case where μ is restricted to one value; that is, we consider only a single channel. There is then only one $\kappa_\mu(\equiv\kappa)$, so that

$$\mathscr{T}_{fi}=\frac{1-e^{2i\delta}}{2\pi i}+e^{2i\delta}\frac{\kappa}{1+i\pi\kappa} \tag{2.48'''}$$

and

$$S_{fi}=e^{2i\delta}\frac{1-i\pi\kappa}{1+i\pi\kappa}=e^{2i(\delta+\sigma)} \tag{2.49}$$

Assuming as an example that there are only two states Φ_1 and Φ_2 in $\mathscr{Q}$, (2.46) yields

$$\kappa=\frac{1}{2\pi}\left(\frac{\gamma_1^2}{E-e_1}+\frac{\gamma_2^2}{E-e_2}\right)$$

and

$$\frac{\kappa}{1+i\pi\kappa}=\frac{1}{2\pi}\frac{\Gamma E-\Lambda}{(E-e_1)(E-e_2)+(i/2)(\Gamma E-\Lambda)} \tag{2.50}$$

where

$$\Gamma\equiv\gamma_1^2+\gamma_2^2 \qquad \Lambda=\gamma_1 e_2+\gamma_2 e_1$$

Equation (2.50) involves only four real numbers, while the resonance series (2.33) for two resonances would involve apparently four complex numbers for the single-channel case. However, as (2.50) demonstrates, there must be two independent relations between these constants. One of these is given by (2.38). The others can be derived either from the properties of the secular equation (2.31) or from the unitarity condition [see Levin and Feshbach (73)].

3. DERIVATION OF THE OPTICAL MODEL POTENTIAL

The optical model potential as originally defined [Feshbach, Porter, and Weisskopf (54)] is that single-channel potential which would generate the energy average of the elastic scattering amplitude (or the transition amplitude). The

energy average was to be taken over a domain, ΔE, which is large compared to both the width Γ of the individual resonances or fluctuations in the cross section as well as their spacing D. On the other hand, ΔE should be small compared to energies Γ_{SP} over which the potential scattering amplitude varies considerably. For nuclear potentials Γ_{SP} is of the order of several MeV, which is to be compared with the width of compound nuclear resonances, which can extend from fractions of an eV to a few hundred kilovolts.

In the context of the formalism of this chapter, the energy averaging of the transition amplitude is equivalent to an energy average of $P\Psi$, since $\mathscr{P}$ contains the incident channel and therefore $\mathscr{T}_{fi}^{(P)}$ contains the elastic scattering amplitude. $\mathscr{P}$ can, of course, contain all the prompt channels, so that the analysis which follows will generalize the optical model to include these prompt processes, such as the single- and multistep direct reactions.

This result for $\langle P\Psi \rangle$ can be seen directly from (2.27′), which can be written as follows:

$$P\Psi_i = \chi_i^{(+)} + \frac{1}{E^{(+)} - H_{PP}} H_{PQ} Q\Psi_i \tag{3.1}$$

Averaging both sides according to conditions outlined in the preceding paragraph yields

$$\langle P\Psi_i \rangle = \chi_i^{(+)} + \frac{1}{E^{(+)} - H_{PP}} H_{PQ} \langle Q\Psi_i \rangle \tag{3.2}$$

since the only quantity on the right side that varies with energy rapidly enough to be changed by the averaging procedure is $Q\Psi$. The transition amplitude generated by (3.2) is clearly the energy average of that generated by (3.1), as can be seen from (2.27):

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \langle \chi_f^{(-)} | H_{PQ} Q\Psi_i \rangle$$

Upon averaging this quantity, one obtains

$$\langle \mathscr{T}_{fi} \rangle = \mathscr{T}_{fi}^{(P)} + \langle \chi_f^{(-)} | H_{PQ} \langle Q\Psi_i \rangle \rangle \tag{3.3}$$

To determine the optical model potential it is necessary to determine the Schrödinger equation satisfied by $\langle P\Psi \rangle$. Toward this end, we replace $Q\Psi$ in (3.2) by the result, (2.30), obtaining

$$\langle P\Psi_i \rangle = \chi_i^{(+)} + \frac{1}{E^{(+)} - H_{PP}} H_{PQ} \left\langle \frac{1}{e_{QQ}} \right\rangle H_{QP} \chi_i^{(+)} \tag{3.4}$$

where

$$e_{QQ} = E - H_{QQ} - W_{QQ} \tag{3.5}$$

Operating on both sides of (3.4) with $E^{(+)} - H_{PP}$ yields

$$(E - H_{PP})\langle P\Psi_i\rangle = H_{PQ}\left\langle \frac{1}{e_{QQ}} \right\rangle H_{QP}\chi_i^{(+)} \tag{3.6}$$

This could have been directly obtained from (2.3) since it leads directly to

$$(E - H_{PP})\langle P\Psi_i\rangle = H_{PQ}\langle Q\Psi_i\rangle \tag{3.7}$$

We now replace $\chi_i^{(+)}$ in favor of $\langle P\Psi\rangle$ by solving (3.4):

$$\chi_i^{(+)} = \frac{1}{1 + [1/(E^{(+)} - H_{PP})]H_{PQ}\langle 1/e_{QQ}\rangle H_{QP}}\langle P\Psi_i\rangle \tag{3.7'}$$

so that (3.6) becomes

$$(E - H_{PP})\langle P\Psi_i\rangle = H_{PQ}\left\langle \frac{1}{e_{QQ}} \right\rangle H_{QP}\frac{1}{1 + [1/(E^{(+)} - H_{PP})]H_{PQ}\langle 1/e_{QQ}\rangle H_{QP}}\langle P\Psi_i\rangle \tag{3.7''}$$

Using the operator identity

$$B\frac{1}{1 + CAB} = \frac{1}{1 + BCA}B$$

with

$$A = \left\langle \frac{1}{e_{QQ}} \right\rangle, \quad B = H_{QP}, \quad \text{and} \quad C = \frac{1}{E^{(+)} - H_{PP}}H_{PQ}$$

one obtains

$$\begin{aligned}(E - H_{PP})\langle P\Psi_i\rangle &= H_{PQ}\left\langle \frac{1}{e_{QQ}} \right\rangle \frac{1}{1 + H_{QP}[1/(E^{(+)} - H_{PP})]H_{PQ}\langle 1/e_{QQ}\rangle} H_{QP}\langle P\Psi_i\rangle \\ &= H_{PQ}\left\langle \frac{1}{e_{QQ}} \right\rangle \frac{1}{1 + W_{QQ}\langle 1/e_{QQ}\rangle} H_{QP}\langle P\Psi_i\rangle\end{aligned}$$

Therefore, the optical model equation for $\langle P\Psi\rangle$ is

$$\left[E - H_{PP} - H_{PQ}\frac{1}{\langle 1/e_{QQ}\rangle^{-1} + W_{QQ}}H_{QP}\right]\langle P\Psi_i\rangle = 0 \tag{3.8}$$

The optical Hamiltonian is therefore

$$H_{\text{opt}} = H_{PP} + H_{PQ} \frac{1}{\langle 1/e_{QQ} \rangle^{-1} + W_{QQ}} H_{QP} \tag{3.9}$$

If $\langle 1/e_{QQ} \rangle$ were replaced by $1/e_{QQ}$, (3.9) would return to the exact form (2.7).

The term $\langle 1/e_{QQ} \rangle$ represents in an average way the impact of the omitted channels in $\mathscr{Q}$. Obviously, the value depends greatly on the choice of the states to be in $\mathscr{P}$, which should be made according to the physics of the situation being considered. Choosing only the elastic channel to be in $\mathscr{P}$ throws all the effects of the other prompt channels into $\langle 1/e_{QQ} \rangle$. This may be what is needed if only elastic scattering and total cross section are of interest. If prompt channels exist and are included in $\mathscr{P}$, (3.8) becomes a set of coupled equations with complex diagonal and coupling potentials. It should not be surprising that the diagonal component of the optical potential for the elastic channel differs from that of the optical potential obtained by restricting $\mathscr{P}$ to the elastic channel. As we shall see, H_{opt} is absorptive. As might be anticipated, the absorption is larger for the single-channel $\mathscr{P}$ space since it must contain the effects of the channels that have been placed in $\mathscr{Q}$.

The reader should note that the optical model potential described in this section differs from the optical model potential derived in Chapter II. The latter is obtained by the multiple scattering approximation valid in the limit of high energies and short wavelengths. The optical model potential of this chapter is a consequence of energy averaging the fine-structure resonance. As a consequence of the averaging process, detailed information (e.g., regarding the resonances) cannot be obtained from the optical model potential and its wave function.

Averaging. It now remains to evaluate $\langle 1/e_{QQ} \rangle$. The average of any function $F(E)$ is given in terms of a normalized density $\rho(E, E_0)$ as follows:

$$\langle F(E) \rangle = \int \rho(E, E_0) F(E_0) \, dE_0 \tag{3.10}$$

where

$$\int \rho(E, E_0) \, dE_0 = 1$$

Therefore,

$$\left\langle \frac{1}{e_{QQ}} \right\rangle = \int \rho(E, E_0) \frac{1}{E_0 - H_{QQ} - W_{QQ}} \, dE_0 \tag{3.11}$$

Note that W_{QQ} is a function of energy but a slowly varying one, so that the only energy dependence we need to consider is the explicit one. It should be

recalled that $1/e_{QQ}$ varies rapidly with energy because of its complex poles, that is, because of resonances or fluctuations.

Two forms of $\rho(E, E_0)$ are in common use:

$$\rho(E, E_0) = \frac{\Delta}{2\pi} \frac{1}{(E - E_0)^2 + (\Delta/2)^2} \qquad \text{Lorentzian} \tag{3.12}$$

$$\rho(E, E_0) = \begin{cases} \dfrac{1}{\Delta E} & |E - E_0| < \dfrac{\Delta E}{2} \\ 0 & |E - E_0| > \dfrac{\Delta E}{2} \end{cases} \qquad \text{box average} \tag{3.13}$$

These two forms emphasize differently the resonances at a distance from E_0, the box average giving them zero weight. It is possible to devise a family of density functions which vary from the Lorentzian to a Gaussian that is similar to the box form [Levin and Feshbach (73)]. These will give differing answers for the integral defining $\langle 1/e_{QQ} \rangle$. For the present it will suffice to discuss the consequences of the two examples above.

The Lorentzian has the advantage of analytic simplicity. Consider the Lorentzian average of $F(E)$:

$$\langle F(E) \rangle_L = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} \frac{F(E_0)}{(E - E_0)^2 + (\Delta/2)^2} dE_0$$

Assuming that $F(E_0)$ has no singularities in the upper half of the complex E plane and that it is well behaved on the infinite semicircle, it is a simple matter to evaluate the integral using the calculus of residues. One obtains

$$\langle F(E) \rangle_L = F\left(E + \frac{i\Delta}{2}\right) \tag{3.14}$$

Hence

$$\left\langle \frac{1}{e_{QQ}} \right\rangle_L = \frac{1}{E + (i\Delta/2) - H_{QQ} - W_{QQ}} \tag{3.15}$$

The effect of the averaging process is to increase the width of each resonance by Δ. Since Δ is assumed to be large compared to each of these widths, the fluctuations caused by the resonances are completely smoothed. The optical Hamiltonian, (3.9), becomes

$$(H_{\text{opt}})_L = H_{PP} + H_{PQ} \frac{1}{E - H_{QQ} + (i/2)\Delta} H_{QP} \tag{3.16}$$

To obtain the box average it is convenient to evaluate a typical term in an expansion of $1/e_{QQ}$:

$$\left\langle \frac{1}{E - E_s + i\Gamma_s/2} \right\rangle = \frac{1}{\Delta E} \int_{E-(1/2)\Delta E}^{E+(1/2)\Delta E} \frac{dE_0}{E_0 - E_s + (i/2)\Gamma_s}$$

$$= \frac{1}{\Delta E} \int_{(2/\Gamma_s)[E-E_s-(1/2)\Delta E]}^{(2/\Gamma_s)[E-E_s+(1/2)\Delta E]} \frac{x - i}{x^2 + 1} dx$$

As long as

$$|E - E_s| \ll \tfrac{1}{2}\Delta E \tag{3.17}$$

that is, near the center of the interval,

$$\left\langle \frac{1}{E - E_s + (i/2)\Gamma_s} \right\rangle = -\frac{i\pi}{\Delta E} \tag{3.18}$$

Condition (3.17) can be rephrased to state that (3.18) is valid for resonances in an interval δE about E as long as $\delta E \ll \Delta E$. For (3.18) to be useful, the following inequalities must hold:

$$\Delta E \gg \delta E \gg \begin{cases} \Gamma \\ D \end{cases} \tag{3.19}$$

This condition is not restrictive except in the case of extraordinarily broad resonances[‡]. From (3.18) it follows that

$$\left\langle \frac{1}{e_{QQ}} \right\rangle = -\frac{i\pi}{\Delta E} \tag{3.19'}$$

The optical Hamiltonian is then

$$(H_{\text{opt}})_{\text{box}} = H_{PP} - \frac{i\pi}{\Delta E} H_{PQ} \frac{1}{1 - (i\pi/\Delta E)W_{QQ}} H_{QP} \tag{3.20}$$

In both cases the optical Hamiltonian is complex. In the limit of large Δ (or ΔE) the imaginary part is negative definite, as it must be so that the optical

[‡]Equation (3.18) is not valid for resonances near the edge of the interval ΔE. This is referred to as the *edge effect*. If the edge effect is substantial one expects a sensitivity with respect to the averaging interval ΔE. Generally, experimental results are quoted when the data are insensitive to ΔE and thus (3.18) may be taken as valid for that reason. The edge effect is also present for the Lorentzian since it weights these resonances beyond $(E \pm \Delta/2)$ in a particular way.

potential is absorptive. This is expected since the elimination of the channels in $\mathcal{Q}$ space leaves some of the flux unaccounted for.

More graphically, an energy average of the wave function is equivalent to the construction of a wave packet, whose passage time at a given space point is on the order of $(\hbar/\Delta E)$. The differing averaging densities, ρ, develop different shapes and time dependence for the wave packet. The Lorentzian, for example, leads to an exponential decay with time. As developed in deShalit and Feshbach (74, p. 91 and following), the prompt amplitude will not contain any contribution from the delayed component, whose time delay is $\hbar/\Gamma$, $\Gamma \ll \Delta E$. Thus when the incident wave strikes the nucleus, part of it passes through promptly, and part of it is delayed. The prompt amplitude does not contain the latter, so that the averaged prompt amplitude exhibits absorption. Moreover, as the width for the resonances (or fluctuations) increases, or equivalently, if it is necessary to consider a number of resonances simultaneously because they overlap, this separation in time between the delayed and time component will be reduced, raising the question of how much should be assigned to each. Clearly, if the width is on the same order as ΔE, there can be no distinction between the two and the optical potential should be real, in the absence, of course, of true inelastic processes.

The shape of the wave packet plays a critical role for this issue. $(H_{\text{opt}})_L$ does not depend on W_{QQ} and does not show this effect. On the other hand, $(H_{\text{opt}})_{\text{box}}$ does. For that case we evaluate the imaginary part of the expectation value of this potential, which is directly proportional to the absorption:

$$\begin{aligned}&\operatorname{Im}\langle\langle P\Psi\rangle|(H_{\text{opt}})_{\text{box}}\langle P\Psi\rangle\rangle \\ &\quad=\operatorname{Im}\left\langle\langle P\Psi\rangle\left|\frac{-i\pi}{\Delta E}H_{PQ}H_{QP}\frac{1}{1-(i\pi/\Delta E)[1/(E^{(+)}-H_{PP})]H_{PQ}H_{QP}}\langle P\Psi\rangle\right\rangle\right.\end{aligned} \tag{3.21}$$

where we have used (3.7″) and (3.19′). Noting that (3.7′) implies that

$$\langle P\Psi\rangle=\left(1-\frac{i\pi}{\Delta E}\frac{1}{E^{(+)}-H_{PP}}H_{PQ}H_{QP}\right)\chi_i^{(+)}$$

we can rewrite (3.21) as follows:

$$\begin{aligned}&\operatorname{Im}\langle\langle P\Psi\rangle|(V_{\text{opt}})_{\text{box}}\langle P\Psi\rangle\rangle \\ &\quad=\operatorname{Im}\left\langle\chi_i^{(+)}\left|\left(1+\frac{i\pi}{\Delta E}H_{PQ}H_{QP}\frac{1}{E^{(-)}-H_{PP}}\right)\left(\frac{-i\pi}{\Delta E}H_{PQ}H_{QP}\right)\chi_i^{(+)}\right\rangle\right.\end{aligned}$$

Assuming that $\chi_i^{(+)}$ is an eigenstate of S_P, so that

$$\chi_i^{(+)}=e^{i\delta}F$$

where F is real, one obtains

$$\operatorname{Im}\langle\langle P\Psi\rangle|(V_{\text{opt}})_{\text{box}}\langle P\Psi\rangle\rangle = \left\langle \chi_i^{(+)} \middle| -\frac{\pi}{\Delta E} H_{PQ}H_{QP} + \pi\frac{\pi^2}{\Delta E^2} H_{PQ}H_{QP}\delta(E-H_{PP})H_{PQ}H_{QP}\chi_i^{(+)} \right\rangle$$

Introducing the widths [see (2.35) and (2.22)]

$$\Gamma_s = 2\pi|\langle\Phi_s|H_{QP}\chi_i^{(+)}\rangle|^2$$

and restricting $\mathscr{P}$ space so that it includes only $\chi_i^{(+)}$, we find that

$$\text{absorption} \sim -\operatorname{Im}\langle\langle P\Psi\rangle|(V_{\text{opt}})_{\text{box}}\langle P\Psi\rangle\rangle \leqslant \left[\frac{1}{2\Delta E}\sum\Gamma_s - \pi\left(\frac{1}{2\Delta E}\right)^2(\sum\Gamma_s)^2\right] \tag{3.22'}$$

Thus the absorption differs from zero if

$$\frac{1}{\Delta E}\sum\Gamma_s \equiv \frac{\langle\Gamma\rangle}{\langle D\rangle} \leqslant \frac{2}{\pi} \tag{3.22}$$

where $\langle D\rangle$ is the average spacing between resonances. This condition agrees in order of magnitude with Wigner's limit [Lane, Thomas, and Wigner (55)] and has been derived in another fashion [Feshbach (69)]. This should not be taken as a rigorous condition on a given channel strength function, as it depends on a particular choice of the averaging density. However, it does have a qualitative significance. In fact, it can be shown [Mello and Feshbach (72)] that large widths imply a strong correlation of widths with resonance energies and a hypersensitivity with regard to the choice of the averaging interval ΔE.

4. INTERMEDIATE STRUCTURE, DOORWAY STATE RESONANCES, AND GIANT RESONANCES[‡]

The subjects of this section have been discussed in deShalit and Feshbach (74, pp. 99–104) and in Section I.6. It will therefore suffice to provide only a brief review before entering into the formal description.

Intermediate structure refers to an energy dependence of the cross section whose scale, Γ_d, lies between the width, Γ_{CN}, for the fine structure produced by compound nuclear resonances or Ericson–Brink fluctuations (see Chapter IV) and the width, Γ_{SP}, for the gross structure that would appear in the cross section

[‡]Block and Feshbach (63); Feshbach, Kerman, and Lemmer (67).

for single-channel elastic scattering produced by a real local potential, that is,

$$\Gamma_{SP} \gg \Gamma_d \gg \Gamma_{CN} \tag{4.1}$$

To observe intermediate structure it is necessary to average the cross section (either by numerically averaging good resolution data or by using poorer, but not too poor, experimental energy resolution). In this way the fluctuations in the cross section are smoothed and larger widths become visible. The interval ΔE over which the average is made should be less than Γ_d but much greater than D, the spacing for compound nuclear levels. This process is demonstrated in Fig. I.13.7 of deShalit and Feshbach (74) for an isobar analog resonance. Conversely, intermediate structure should exhibit fine structure when examined with sufficiently good energy resolution. This is illustrated by Fig. I.1.1, which shows the fine structure associated with two isobar analog resonances at 2.77 and 3.14 MeV.

Many examples of intermediate structure have been observed. So far we have discussed the giant electric dipole resonance, the subthreshold fission cross section, and the isobar analog resonance [see deShalit and Feshbach (74, p. 99–104)]. Giant resonances generally are examples of intermediate structure. The quadrupole resonance at an excitation energy of $60/A^{1/3}$ MeV, the electric monopole mode, and the Gamow–Teller resonances are examples. The light heavy-ion system $^{12}C + {}^{12}C$ exhibits intermediate structure involving many resonances, as shown in Fig. 4.1. These fragment into fine structure when examined with very high energy resolution.

All the foregoing cases involve isolated resonances. However, there will be cases in which they overlap. For these a statistical theory of the type that has been discussed briefly in Section I.6 and is discussed more fully in Chapter IV, is appropriate.

The states corresponding to these resonances have been referred to as *doorway states* and the corresponding resonances as *doorway state resonances* [Block and Feshbach (63)], to emphasize that they serve as the first stage beyond the entrance channel in the development of the complex compound nuclear state; that is, the system would have to pass through the doorway state before the full complexity of the compound nuclear state could be generated. This is illustrated in Fig. 4.2. The doorway state is a relatively simple state, not as complex as the compound state or as simple as the entrance channel state paralleling the inequality (4.1). The hypothesis that such simple states exist and that they are the only states that couple strongly to the entrance channel will be referred to as the *doorway state hypothesis.*

A simple, although somewhat idealized example will help to make the doorway state hypothesis more concrete. Suppose that the incident projectile is a nucleon. Suppose, moreover, that its energy is such that no reactions are possible, so that only elastic scattering can occur. The entrance channel wave function, $\chi_i^{(+)}$, then describes the motion of the incident nucleon in the field of the target nucleus. This nucleon–nuclear interaction is specified by a model Hamiltonian that is appropriate for this energy range. The remainder of the

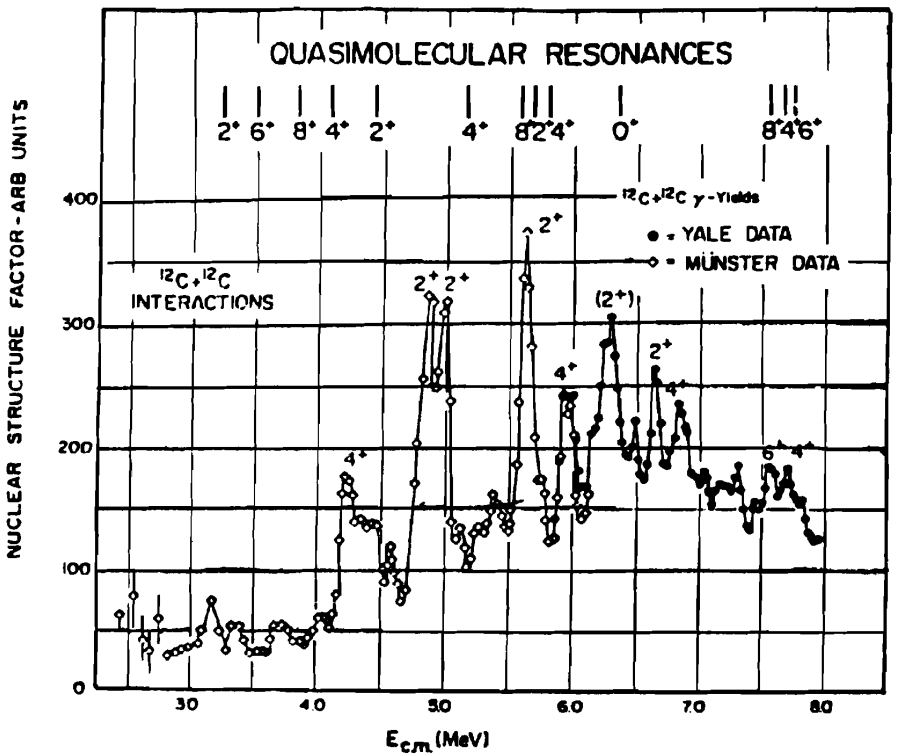

FIG. 4.1. Nuclear structure factors derived from the total γ-radiation yields of the $^{12}C+^{12}C$ interaction. [From Erb and Bromley (84).]

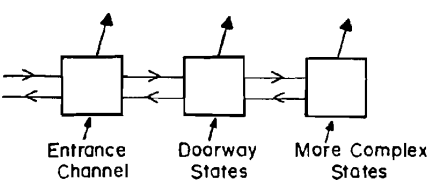

FIG. 4.2

wave function is generated by the residual Hamiltonian given by the difference between the exact and model Hamiltonians. As an example, suppose that we idealize this residual interaction by assuming that it consists of a sum of two-body interactions. Furthermore, as an example, suppose that the target nucleus has a closed shell or subshell so that the appropriate entrance channel Hamiltonian is the shell model Hamiltonian. The entrance channel is then a one-particle state in shell model terminology. The residual interaction acting on $\chi_i^{(+)}$ will excite a particle hole pair, the incident nucleon changing its state, generating a two-particle/one-hole state (2p–1h). The states produced in this fashion, or a linear combination of them, are doorway states since they are the

only ones that are coupled by the residual interaction to the entrance channel. All the states of this complexity (i.e., of the 2p–1h structure form the doorway state space, $\mathscr{D}$. Since the particle–hole interaction can build up collective states with preferred spin and isospin (e.g., the dipole $J=1^-$, $T=1$ state), a doorway state may often be more simply described as a particle plus vibration state. More generally, a doorway state can be a particle plus a collective state, and then it may be more convenient to employ collective variables in describing the residual interaction.

The above is simply an example; the doorway concept is not limited to the case of a nucleon incident on a shell model target nucleus. The examples of intermediate structure listed earlier include the case of an incident photon (El), an incident α-particle, or the collision of two ^{12}C nuclei. What is required is that upon the first collision [Weisskopf (60)] the residual interaction produces a relatively simple configuration, analogous to the 2p–1h states discussed above. In the case of the giant dipole resonance, that simple state is well known to be a linear combination of particle–hole excitations [see deShalit and Feshbach (74, p. 491)].

As discussed in the introduction to this chapter, a resonance occurs whenever there is an approximate state of the system that is bound and whose energy equals the total energy of the system. The important point is that in this state the energy has been redistributed and a sufficient fraction of the incident kinetic energy has become internal energy. A doorway state has just this property, namely that some of the system's energy has been used for excitation rather than remaining as kinetic energy. In the example quoted above, this energy is used to excite a particle–hole pair. If, in addition, the total energy of the doorway state equals the incident kinetic energy, a doorway state resonance will occur, producing as a consequence intermediate structure in the cross section.

The resonant doorway state wave function will generally be composed not only of the states generated directly by the residual interaction acting on the entrance channel wave function but will also include additional components of the same complexity. These are contained within doorway space $\mathscr{D}$. In our example, after the initial formation of 2p–1h states, the nuclear Hamiltonian acting on them will produce others (as well as states of different complexity) which are not directly coupled to the entrance channel. The resonant doorway state, being an approximate eigenstate, will generally involve these components as well as the directly coupled ones.

The resonant doorway state is not an exact stationary eigenstate of the nuclear Hamiltonian. It will decay with time and thus has a width. In addition to the *escape width* $\Gamma_d^{\uparrow}$, caused by transitions to various exit channels and common to all types of resonances, there is the width, which is a consequence of the possible transition to states of higher complexity, roughly speaking to the fine structure states (see Fig. 4.2). This width, $\Gamma_d^{\downarrow}$, is referred to as the *spreading width*, $\Gamma_d^{\downarrow}$. The total doorway state width, Γ_d, is the sum of these two:

$$\Gamma_d = \Gamma_d^{\downarrow} + \Gamma_d^{\uparrow} \tag{4.2}$$

The visibility of the doorway state resonance depends critically upon Γ_d. If Γ_d is too large, intermediate structure will not be seen.

To see this, let us turn to the expression for the $l = 0$ elastic scattering cross section, which applies to this case [compare (2.20)].

$$\sigma = \frac{4\pi}{k^2}\left|\sin\delta - e^{i\delta}\frac{\Gamma_d^{\uparrow}/2}{(E - E_d) + (i/2)(\Gamma_d^{\downarrow} + \Gamma_d^{\uparrow})}\right|^2$$

where $\hbar^2k^2/2m$ is the energy and δ is a phase shift. We see that $\Gamma_d^{\downarrow}$ acts as an additional absorption. The maximum magnitude of the resonant doorway term, the second one, occurs at $E = E_d$ and is equal to $\Gamma_d^{\uparrow}/\Gamma_d$, proving the point made in the preceding paragraph.

Stating this result in another way, if the coupling between the resonant doorway state and the compound nuclear states, whose energies differ appreciably from E_d, is substantial, the spreading width $\Gamma_d^{\downarrow}$ will be large and the resonant cross section will be reduced appreciably. Moreover, if as a consequence of this coupling, Γ_d is on the order of or greater than D_d, the energy spacing between doorway resonances, these will overlap. The doorway resonance will not be isolated and an energy average of the fine structure will consequently average over more than one doorway state, preventing direct observation of the character of each individual doorway resonance.

We turn to the quantitative description of doorway state phenomena. The Hilbert space will now be partitioned into three parts, $\mathscr{P} + \mathscr{Q} + \mathscr{Q}'$, where $\mathscr{P}$ will contain the prompt channels, $\mathscr{Q}$, the doorway states, and $\mathscr{Q}'$ the more complex states, as indicated by Fig. 4.2. Comparing with (2.1), we have

$$\mathscr{Q} = \mathscr{D} + \mathscr{Q}' \tag{4.3}$$

It is, however, more convenient, for the derivations given below, to group the doorway states with the prompt channels. Let the projection operators that project onto the $\mathscr{P}, \mathscr{D}$, and $\mathscr{Q}'$ subspaces be P, D, and Q', respectively. Orthogonality of the spaces is presumed, so that

$$P^2 = P \qquad PD = 0 \qquad PQ' = 0 \qquad DQ' = 0$$
$$P + D + Q' = 1 \tag{4.4a}$$

Finally, let

$$P' \equiv P + D \qquad P' + Q' = 1 \tag{4.4b}$$

The analysis of Section 2 applies. One can, for example, immediately write

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P')} + \left\langle \phi_f^{(-)} \middle| H_{P'Q'} \frac{1}{E^{(+)} - H_{Q'Q'} - W_{Q'Q'}} H_{Q'P'} \middle| \phi_i^{(+)} \right\rangle \tag{4.5}$$

where

$$(E - H_{P'P'})\phi = 0$$

The functions ϕ can be resolved into prompt and doorway components $P\phi$ and $D\phi$ satisfying the equations

$$(E - H_{PP})P\phi = H_{PD}D\phi \tag{4.6}$$

$$(E - H_{DD})D\phi = H_{DP}P\phi \tag{4.7}$$

$\mathcal{T}_{fi}^{(P')}$ is the transition amplitude generated by this system. Since these equations are identical in form (simply replace Q by D) with (2.3) and (2.4) it is entirely possible, depending of course on H, for this system to exhibit resonances which will be present in $\mathcal{T}_{fi}^{(P')}$ (see Fig. 4.3). The analysis follows exactly that leading to (2.20) or (2.30′). These resonances are the progenitors of the doorway state resonances, for the coupling to $\mathscr{Q}'$ has not yet been included. They will have a width that is roughly the escape width $\Gamma^{\uparrow}$. To obtain the full width it is necessary to average over the fine structure, that is, obtain the optical potential for this case.

As we saw earlier, the energy averaged wave function $\langle P'\Psi\rangle$ satisfies

$$(E - H_{\text{opt}})\langle P'\Psi\rangle = 0$$

where

$$H_{\text{opt}} = H_{P'P'} + H_{P'Q'}\frac{1}{\langle 1/e_{Q'Q'}\rangle^{-1} + W_{Q'Q'}}H_{Q'P'} \tag{4.8}$$

We now introduce the critically important assumption, which we shall refer to as the *strong doorway assumption*, namely that $\mathscr{P}$ couples *only* to $\mathscr{Q}$. In other words, $\mathscr{Q}'$ does not couple directly to $\mathscr{P}$. This assumption leads to the equations

$$H_{PQ'} = 0 \qquad H_{P'Q'} = H_{DQ'} \tag{4.9}$$

Therefore,

$$H_{\text{opt}} = H_{P'P'} + H_{DQ'}\frac{1}{\langle 1/e_{Q'Q'}\rangle^{-1} + W_{Q'Q'}}H_{Q'D} \tag{4.10}$$

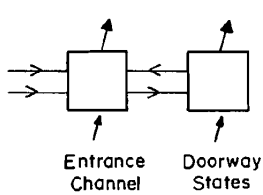

FIG. 4.3

It follows that the equation for $\langle P'\Psi\rangle$ can, by virtue of

$$\langle P'\Psi\rangle = \langle P\Psi\rangle + \langle D\Psi\rangle$$

be written as a pair of coupled equations:

$$(E - H_{PP})\langle P\Psi\rangle = H_{PD}\langle D\Psi\rangle \tag{4.11}$$

$$(E - H_{DD} - \bar{W}_{DD})\langle D\Psi\rangle = H_{DP}\langle P\Psi\rangle \tag{4.12}$$

where

$$\bar{W}_{DD} = H_{DQ'}\frac{1}{\langle 1/e_{Q'Q'}\rangle^{-1} + W_{Q'Q'}}H_{Q'D} \tag{4.13}$$

Comparing with (4.7), we see that the coupling of $\mathscr{D}$ with $\mathscr{Q}'$ has produced, upon energy averaging over the fine structure, an additional complex term $\bar{W}_{DD}$ whose imaginary part will give rise to $\Gamma_d^{\downarrow}$. One can now immediately obtain the analog of (2.30′) by the procedure used to derive that equation. The result is

$$\langle \mathscr{T}_{fi}\rangle = \mathscr{T}_{fi}^{(P)} + \left\langle \chi_f^{(-)}\middle| H_{PD}\frac{1}{E - H_{DD} - \bar{W}_{DD} - W_{DD}}H_{DP}\chi_i^{(+)}\right\rangle \tag{4.14}$$

where $\chi_{f,i}^{(\pm)}$ are, as before, solutions of

$$(E - H_{PP})\chi = 0$$

and

$$W_{DD} = H_{DP}\frac{1}{E^{(+)} - H_{PP}}H_{PD} \tag{4.15}$$

W_{DD} describes the coupling of the doorway states to the entrance and exit channels.

If one now makes the single isolated doorway state assumption, (4.14) becomes

$$\langle \mathscr{T}_{fi}\rangle = \mathscr{T}_{fi}^{(P)} + \frac{\langle \chi_f^{(-)}|H_{PD}\psi_d\rangle\langle\psi_d|H_{DP}\chi_i^{(+)}\rangle}{E - E_d + (1/2)i(\Gamma_d^{\uparrow} + \Gamma_d^{\downarrow})} \tag{4.16}$$

where

$$\begin{aligned} E_d &= \mathrm{Re}\langle\psi_d|(H_{DD} + W_{DD} + \bar{W}_{DD})\psi_d\rangle \\ \Gamma_d^{\uparrow} &= -2\,\mathrm{Im}\langle\psi_d|W_{DD}\psi_d\rangle = 2\pi\sum_{\gamma}|\langle\chi_\gamma^{(-)}|H_{PD}\psi_d\rangle|^2 \end{aligned} \tag{4.17}$$

$$\Gamma_d^{\downarrow} = -2\,\mathrm{Im}\langle\psi_d|\bar{W}_{DD}\psi_d\rangle = \frac{4}{\Delta}\sum_q \frac{|\langle\psi_d|H_{DQ}\Phi_q\rangle|^2}{1+[2(E-\mathscr{E}_q)/\Delta]^2}$$

$$\to \frac{4}{\Delta}\sum_q |\langle\psi_d|H_{DQ}\Phi_q\rangle|^2$$

Here ψ_d is the doorway state wave function and Φ_q are the wave functions describing the fine structure resonances in $\mathscr{Q}'$ [see (2.9)]. The expression for $\Gamma_d^{\downarrow}$ has been obtained by using the Lorentzian average for $\langle 1/e_Q\rangle$. The wave function ψ_d, a solution of

$$(E - H_{DD})\psi_d = 0$$

is a bound-state wave function. The apparent difference between the second and third line of (4.17) is a consequence of the differing normalizations of ψ_d, which is normalized so that the volume integral of $|\psi_d|^2$ is unity, and $\chi_f^{(-)}(E)$, which is normalized so that volume integral of $\chi_f^{(-)*}(E)\chi_f^{(-)}(E')$ is $\delta(E-E')$. As anticipated in the introductory remarks, the magnitude of $\Gamma_d^{\downarrow}$ will depend on the magnitude of the residual interaction $H_{DQ'}$, and on the rate with which the series over q converges. If contributions from distant resonances are important, $\Gamma_d^{\downarrow}$ will tend to be large and the resonance obscured by the single-step amplitude $\mathscr{T}_{fi}^{(P)}$.

Some further consequences of doorway state resonances can be uncovered by studying the one-channel case. Then the elastic scattering $\langle\mathscr{T}\rangle$ matrix, according to (4.16) and (4.17), is given by

$$\langle\mathscr{T}\rangle = \mathscr{T}^{(P)} + \frac{1}{2\pi}e^{2i\delta}\frac{\Gamma_d^{\uparrow}}{(E-E_d)+\frac{1}{2}i(\Gamma_d^{\downarrow}+\Gamma_d^{\uparrow})}$$

The average S matrix that follows is

$$\langle S\rangle = e^{2i\delta}\frac{(E-E_d)+\frac{1}{2}i(\Gamma_d^{\downarrow}-\Gamma_d^{\uparrow})}{(E-E_d)+\frac{1}{2}i(\Gamma_d^{\downarrow}+\Gamma_d^{\uparrow})} \tag{4.18}$$

As is obvious, $\langle S\rangle$ is not unitary because of the spreading width, $\Gamma_d^{\downarrow}$, which acts, as stated earlier, as an absorption. The transmission factor (p. 243), T, reflects the presence of absorption since

$$T = 1 - |\langle S\rangle|^2 \tag{4.19}$$

which, in the present case, is

$$T = \frac{\Gamma_d^{\downarrow}\Gamma_d^{\uparrow}}{(E-E_d)^2+\frac{1}{4}\Gamma_d^2} \tag{4.20}$$

Recall that the average absorption cross section is proportional to T. Thus we

obtain the important result that the cross section for absorption into compound nuclear state is enhanced by the presence of doorway state achieving a maximum at $E = E_d$. Obviously, the transmission by the doorway mechanism will be zero if there is no coupling of the entrance channel to the doorway state ($\Gamma_d^{\uparrow} = 0$) or if that state does not couple to the compound nuclear resonances ($\Gamma_d^{\downarrow} = 0$).

The symmetry of T about $E = E_d$ is a consequence of the strong doorway assumption, namely that the entrance channel couples only with the doorway state and not with the more complex states. We now examine the effect of allowing such direct coupling. Upon energy averaging, one obtains, instead of (4.11) and (4.12), the following equations:

$$(E - H_{PP} - \bar{W}_{PP})\langle P\Psi\rangle = (H_{PD} + \bar{W}_{PD})\langle D\Psi\rangle \tag{4.21}$$

$$(E - H_{DD} - \bar{W}_{DD})\langle D\Psi\rangle = (H_{DP} + \bar{W}_{DP})\langle P\Psi\rangle \tag{4.22}$$

where $\bar{W}_{DD}$ is given by (4.13) and

$$\bar{W}_{PD} = H_{PQ'} \frac{1}{\langle 1/e_{Q'Q'}\rangle^{-1} + W_{Q'Q'}} H_{Q'D} \tag{4.23}$$

$\bar{W}_{PD}$ represents the effect of the direct coupling of the subspace $\mathscr{P}$ of the prompt channels and the subspace $\mathscr{Q}'$ on the generation of components contained in subspace $\mathscr{Q}$, while

$$\bar{W}_{PP} = H_{PQ'} \frac{1}{\langle 1/e_{Q'Q'}\rangle^{-1} + W_{Q'Q'}} H_{Q'P} \tag{4.24}$$

represents the effect on the prompt channel of such coupling. All the elements of the $\bar{W}$ matrix are complex. The analysis of (4.21) and (4.22) is identical with that of (4.11) and (4.12).

One effect of the complex $\bar{W}$ matrix is to make the potential scattering $\mathscr{T}$, absorptive, that is, the phase shift for the single-channel case has a positive imaginary part, while another is that the escape width $\Gamma_d^{\uparrow}$ given by

$$\Gamma_d^{\uparrow} = 2\pi\langle\chi_f^{(+)}|(H_{PD} + \bar{W}_{PD})|\psi_d\rangle\langle\psi_d|(H_{DP} + \bar{W}_{DP})\chi_i^{(+)}\rangle \tag{4.25}$$

is complex since $\bar{W}_{PD}$ is not Hermitian; hence

$$\Gamma_d^{\uparrow} = |\Gamma_d^{\uparrow}|e^{i\phi} \tag{4.26}$$

The S-matrix equation (4.18) is replaced by

$$\langle S\rangle = e^{2i\delta}e^{-2\eta}\left[1 - i\frac{|\Gamma_d^{\uparrow}|e^{i\phi}}{(E - E_d) + (i/2)\bar{\Gamma}_d}\right] \tag{4.27}$$

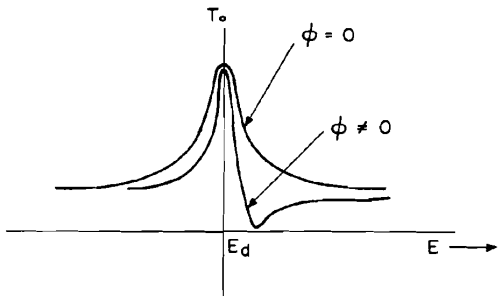

FIG. 4.4

where $\delta + i\eta$ is the potential scattering phase shift and Γ_d is the total width. The transmission T given by (4.19) can readily be evaluated. As long as ϕ is not equal to zero, the resulting T is not symmetric about $E = E_d$. In Fig. 4.4 the T for the symmetric case ($\phi = 0$) is compared with the result obtained when $\phi \neq 0$. The latter is asymmetric and has an interference minimum. Both of these features are a consequence of the relaxation of the strong doorway assumption, $H_{PQ'} = 0$, thereby permitting a direct coupling of $\mathscr{P}$ and $\mathscr{Q}'$, $H_{PQ'} \neq 0$.

According to the preceding discussion, the wave function for the nuclear system Ψ, can be broken up into three components as follows:

$$\Psi = P\Psi + D\Psi + Q'\Psi$$

The first term, $P\Psi$, describes the prompt channels. In the neighborhood of a compound nuclear resonance, $(D + Q')\Psi = Q\Psi$ is given by Φ_s and $D\Psi$ is given by the doorway state, ψ_d. Interestingly, the overlap of ψ_d with Φ_s, $\langle \Phi_s | \psi_d \rangle$, is quite small. In other words, ψ_d need not be a major component of Ψ in order to serve as a doorway state. Or, conversely, the doorway state is built up out of small fractions of the compound nuclear resonance wave functions.

The overlap, $\langle \Phi_s | \psi_d \rangle$, can be related to experimentally observable quantities as follows. The compound nuclear state width Γ_s is related to Φ_s according to (2.22) by

$$\Gamma_s = 2\pi |\langle \chi^{(+)} | H_{PQ} \Phi_s \rangle|^2$$

where for simplicity the discussion is restricted to the single-channel case. Expand Φ_s in terms of ψ_d and the states Φ'_s in $\mathscr{Q}'$. The component of the matrix elements involving Φ'_s is zero because of the strong doorway state assumption. Therefore,

$$\Gamma_s = 2\pi |\langle \Phi_{sd} | \psi_d \rangle|^2 |\langle \chi^{(+)} | H_{PD} \psi_d \rangle|^2$$

or

$$\Gamma_s = \Gamma_d^{\uparrow} |\langle \Phi_{sd} | \psi_d \rangle|^2 \tag{4.28}$$

where the subscript s_d indicates that the compound nuclear resonance is associated with the doorway state ψ_d. The overlap is then

$$|\langle \Phi_{s_d} | \psi_d \rangle|^2 = \frac{\Gamma_s}{\Gamma_d^{\uparrow}} \tag{4.29}$$

Since the doorway state width, Γ_d, encompasses many fine structure resonances and Γ_d and $\Gamma_d^{\uparrow}$ are usually of the same order of magnitude, $\Gamma_s/\Gamma_d^{\uparrow}$ is found to be much smaller than 1. We thus have demonstrated that the overlap $\langle \Phi_s | \psi_d \rangle$ is small.

Another insight can be obtained from the average value of Γ_s averaged now over an interval ΔE containing not only many compound nuclear resonances, but also many doorway states. This ΔE would be appropriate for the optical model if ΔE satisfies $\Gamma_{SP} \gg \Delta E \gg \Gamma_d$. Thus from (4.28)

$$\frac{1}{\Delta E}\sum_s \Gamma_s = \frac{1}{\Delta E}\sum \Gamma_d^{\uparrow} \sum_{s_d} |\langle \Phi_{s_d} | \psi_d \rangle|^2 = \frac{1}{\Delta E}\sum \Gamma_d^{\uparrow}$$

or

$$\frac{\langle \Gamma_s \rangle}{D_s} = \frac{\langle \Gamma_d^{\uparrow} \rangle}{D_d} \tag{4.30}$$

where D_s is the energy spacing of the compound nuclear resonances and D_d that of the doorway states. As usual, the angular brackets indicate average quantities. This equation has a simple interpretation. The ratio $\langle \Gamma \rangle / D$, referred to as the *strength function*, represents the fraction of an energy interval occupied by resonances of average width $\langle \Gamma \rangle$. Thus (4.30) states that this fraction of an energy interval is independent of whether one does an extensive average leading directly to the optical model result given on the left-hand side of the equation or performs the average in two steps in which first $\Delta E \ll \Gamma_d$ and therefore reveals the doorway state resonances and then averages over many doorway states.

Problem. Derive (4.30) by equating the optical model average of the single-channel S matrix with the average of $\langle S \rangle$ [given by (4.18)] averaged over many doorway states.

Returning to (4.29) for the overlap probability, we see that it can be rewritten approximately (since the average of a ratio is not the ratio of the average)

$$\langle |\langle \Phi_s | \psi_d \rangle|^2 \rangle \simeq \frac{D_s}{D_d} = \frac{\rho_d}{\rho_s} \tag{4.31}$$

where ρ gives the density of states denoted by the subscripts. Again since the density of compound nuclear levels is of course much greater than the density

of doorway states, the average overlap probability is much smaller than 1, reaffirming the earlier conclusion.

A. Exit Doorways

The discussion has so far emphasized the connection between the entrance channel and the doorway state. Clearly, if in a reaction the final state is well defined experimentally, there is the possibility that a doorway state exists which bears a similar relation to the exit channel. We shall refer to such doorway states as exit doorways, to distinguish from entrance doorways, which are the doorway states discussed above. As one can see from (4.16), the final state $\chi_f^{(-)}$ is connected to ψ_d in precisely the same way as the initial state $\chi_i^{(+)}$. But there is the possibility that the branching ratio, $|\langle \chi_f^{(-)}|H_{PD}\psi_d\rangle|^2/|\langle \psi_d|H_{DP}\chi_i^{(+)}\rangle|^2$, to the final state is very small, indicating that, in fact, ψ_d is not a doorway state for $\chi_f^{(-)}$. However, there is also the possibility that $\chi_f^{(-)}$ connects strongly to another doorway state $\psi_{d'}$. If these two corresponding resonances overlap, the reaction from $\chi_i^{(+)}$ to $\chi_f^{(-)}$ will be enhanced. This mechanism is important for radiative neutron capture.

One can obtain the transition amplitude for this case starting from the general expression given by (4.14),

$$\langle \mathcal{T}_{fi}\rangle = \mathcal{T}_{fi}^{(P)} + \left\langle \chi_f^{(-)}\middle|\left(H_{PD}\frac{1}{E - H_{DD} - \bar{W}_{DD} - W_{DD}}H_{DD}\right)\chi_i^{(+)}\right\rangle \tag{4.32}$$

When there is more than one isolated doorway state it becomes necessary to diagonalize $(H_{DD} + \bar{W}_{DD} + W_{DD})$, a procedure paralleling that described after (2.32). Let the eigenvalues of this operator be $E_d - i/2\Gamma_d$, the corresponding eigenfunction, Ω_d, and those of the adjoint equation $\Omega_d^{(A)}$. Then

$$\langle \mathcal{T}_{fi}\rangle = \mathcal{T}_{fi}^{(P)} + \sum_d \frac{\langle \chi_f^{(-)}|H_{PD}\Omega_d\rangle\langle \Omega_d^{(A)}|H_{DP}\chi_i^{(+)}\rangle}{E - E_d + (i/2)\Gamma_d} \tag{4.33}$$

Sum rules similar to (2.38) can also be derived for this case. In particular,

$$\sum_d \langle \chi_f^{(-)}|H_{PD}\Omega_d\rangle\langle \Omega_d^{(A)}|H_{DP}\chi_i^{(+)}\rangle = \sum_d \langle \chi_f^{(-)}|H_{PD}\psi_d\rangle\langle \psi_d|H_{DP}\chi_i^{(+)}\rangle \tag{4.34}$$

and

$$\tfrac{1}{2}\sum_d \Gamma_d = -2\pi \sum_d \mathrm{Im}\langle \psi_d|\bar{W}_{DD} + W_{DD}|\psi_d\rangle \tag{4.35}$$

We see that these results are very similar in form to that obtained for overlapping compound nuclear resonances, with the difference that there is an additional absorption measured by $\sum_d \mathrm{Im}\langle \psi_d|\bar{W}_{DD}|\psi_d\rangle$. Note also that Ω_d will

be a linear combination of the various ψ_d states so that even if $\langle \chi_f^{(-)} | H_{PD} \psi_i \rangle$, for example, is small, this is not necessarily the case for $\langle \chi_f^{(-)} | H_{PD} \Omega_i \rangle$. In any event, it is now possible because of interference between two terms, one corresponding to an entrance doorway and another to an exit doorway, to obtain an overlap between entrance and exit doorways.

B. Effect on Fine Structure

We have not yet discussed the impact of the doorway state resonance on the fine-structure resonances or, for that matter, the corresponding effect of direct processes on the doorway state resonance. These will be most clearly seen in the widths of these resonances.

Recall that the width of a fine-structure resonance in $\mathcal{Q}'$ space is, in virtue of the partition,

$$P' + Q' = 1$$

given by (2.22), with P and Q replaced by P' and Q', and χ by ϕ of (4.5). For the single-channel case to which the discussion will be limited, the width is given by

$$\Gamma_s = 2\pi |\langle \Phi_s' | H_{Q'P'} \phi_i^{(+)} \rangle|^2 \tag{4.36}$$

where Φ_s' satisfies

$$(\mathscr{E}_s' - H_{Q'Q'}) \Phi_s' = 0 \tag{4.37}$$

Because of the doorway state assumption, (4.36) can be replaced by

$$\Gamma_s = 2\pi |\langle \Phi_s' | H_{Q'D} D\phi_i^{(+)} \rangle|^2 \tag{4.38}$$

To obtain $D\phi_i^{(+)}$, we turn to the coupled equations (4.6) and (4.7). Solving the first of these for $P\phi$, and inserting the result in the second yields

$$(E - H_{DD} - W_{DD}) D\phi = H_{DP} \chi_i^{(+)} \tag{4.39}$$

so that

$$D\phi_i^{(+)} = \frac{1}{E - H_{DD} - W_{DD}} H_{DP} \chi_i^{(+)} \tag{4.40}$$

We now make the additional assumption that there is only one doorway state in the energy domain being considered. Then

$$D\phi_i^{(+)} = \frac{1}{E - \mathscr{E}_d + \frac{1}{2} i \Gamma_d^{\uparrow}} \psi_d \langle \psi_d | H_{DP} \chi_i^{(+)} \rangle$$

Substituting in (4.38) yields

$$\Gamma_s = \frac{2\pi |\langle \Phi_s' | H_{Q'D} \psi_d \rangle|^2 \, |\langle \psi_d | H_{DP} \chi_i^{(+)} \rangle|^2}{(E - \mathscr{E}_d)^2 + \frac{1}{4}(\Gamma_d^{\uparrow})^2}$$

From (4.17)

$$\Gamma_{di}^{\uparrow} = 2\pi |\langle \psi_d | H_{DP} \chi_i^{(+)} \rangle|^2 \tag{4.41}$$

We also introduce the definition

$$\Gamma_{sd}^{\downarrow} \equiv 2\pi \langle \rho_s \rangle |\langle \Phi_s' | H_{Q'D} \psi_d \rangle|^2 \tag{4.42}$$

where $\langle \rho_s \rangle$ is the average level density of the fine structure levels.

Finally, one obtains

$$\Gamma_s(E) = \frac{\langle D_s \rangle}{2\pi} \frac{\Gamma_{sd}^{\downarrow} \Gamma_{di}^{\uparrow}}{(E - \mathscr{E}_d)^2 + \frac{1}{4}(\Gamma_d^{\uparrow})^2} \tag{4.43}$$

The energy dependence of the numerator of this expression is slow, being on the scale of the direct prompt processes represented by $\chi_i^{(+)}$. Within the doorway state resonance the main energy dependence is therefore carried by the denominator. We see that $\Gamma_s(E)$ has its maximum value at $\mathscr{E}_d$, which by the way does not equal the doorway state resonant energy, E_d, because the effect of $\bar{W}_{DD}$ is not included, as it need not be since no averaging has been carried out in obtaining (4.42). (However, if the box average is used, E_d will equal $\mathscr{E}_d$.) Assuming that the difference $(E_d - \mathscr{E}_d)$ is not large compared to $\Gamma_d^{\uparrow}$, one may conclude that for the case of *isolated* fine-structure resonances, those that occur near the center of the doorway state will be broader. These conclusions are modified if the fine-structure resonances are overlapping.

The strong doorway state assumption plays a central role in the foregoing development. If it is relaxed [see the discussion accompanying (4.22)], the general results obtained above would remain, that is, that the widths would be larger when the fine-structure resonance occurs within the doorway state resonance.

This analysis can be adapted for the description of similar relations between the single-particle resonance and both the doorway state and fine-structure resonances. The doorway state case will be considered first.

We divide the space $\mathscr{P}$ into two orthogonal subspaces, $\mathscr{M}$ and $\mathscr{N}$. The single-particle resonance is taken to be the consequence of a bound state in $\mathscr{N}$, coupling to the open channel space $\mathscr{M}$. Moreover, we require that only $\mathscr{N}$ and not $\mathscr{M}$ can couple to $\mathscr{D}$. It is now possible to set up a one-to-one relationship between the corresponding projection operators M, N, and D with those used

to describe the doorway structure, P, D, and Q' [see (4.4)]:

$$\begin{aligned} M &\leftrightarrow P \\ N &\leftrightarrow D \\ D &\leftrightarrow Q' \end{aligned} \tag{4.44}$$

It follows that the bound single-particle state χ_{SP} with energy E_{SP} satisfies

$$(E_{SP} - H_{NN} - W_{NN})\chi_{SP} = 0 \tag{4.45}$$

where

$$W_{NN} \equiv H_{NM} \frac{1}{E^{(+)} - H_{MM}} H_{MN} \tag{4.46}$$

The analog of $\chi^{(+)}$, the solution of the equation

$$(E - H_{PP})\chi^{(+)} = 0$$

is

$$(E - H_{MM})\bar{\chi}_i^{(+)} = 0 \tag{4.47}$$

We may now restate (4.43) in terms of M, N, and D, replacing Φ_s' by ψ_d, ψ_d by χ_{SP}, and χ_i by $\bar{\chi}_i^{(+)}$. One obtains

$$\Gamma_d(E) = \frac{D_d}{2\pi} \frac{\Gamma_{d,SP}^{\downarrow} \Gamma_{SP,i}^{\uparrow}}{(E - \mathscr{E}_{SP})^2 + \frac{1}{4}(\Gamma_{SP}^{\uparrow})^2} \tag{4.48}$$

where

$$D_d \Gamma_{d,SP}^{\downarrow} = 2\pi |\langle \psi_d | H_{DN} \chi_{SP} \rangle|^2 \quad \text{and} \quad \Gamma_{SP,i}^{\uparrow} = 2\pi |\langle \chi_{SP} | H_{NM} \bar{\chi}_i^{(+)} \rangle|^2$$

demonstrating that the doorway state width will have a maximum value at $\mathscr{E}_{SP}$ near the single-particle resonance energy.

A second set of relations, similar to (4.48), leads to a formulation of the relationship between the single-particle resonance and an isolated fine-structure resonance, which is close to that obtained by Lane, Thomas, and Wigner (55). The isomorphism used is

$$\begin{aligned} M &\leftrightarrow P \\ N &\leftrightarrow D \\ Q &\leftrightarrow Q' \end{aligned} \tag{4.49}$$

Relation (4.43) now reads

$$\Gamma_s(E) = \frac{\langle D_s \rangle}{2\pi} \frac{\Gamma^{\downarrow}_{s,SP}\Gamma^{\uparrow}_{SP,i}}{(E - \mathscr{E}_{SP})^2 + \frac{1}{4}(\Gamma^{\uparrow}_{SP})^2} \tag{4.50}$$

where

$$\Gamma^{\downarrow}_{s,SP} = 2\pi \langle \rho_s \rangle |\langle \Phi_s | H_{QN} \chi_{SP} \rangle|^2$$

so that the fine-structure resonance width will be greater near the single-particle resonance energy, $\mathscr{E}_{SP}$.

It should be noted that the $\mathscr{E}_{SP}$ and Γ_{SP} of (4.50) are not identical with those of (4.48). Similarly, the fine-structure resonances in space $\mathscr{Q}'$ are not the same as those in space $\mathscr{Q}$, so that the D_s used in (4.56) is not identical with the D_s in (4.43). However, substantial differences are not expected.

One interesting result that can be obtained from (4.50) yields a direct, albeit approximate, evaluation of $\Gamma^{\downarrow}_{s,SP}$. We sum both sides of (4.50) over all Γ_s occurring within the single-particle giant resonance:

$$\sum_s \Gamma_s(E_s) \to \int \frac{dE_s}{\langle D_s \rangle} \Gamma_s(E_s) = \frac{\langle \Gamma^{\downarrow}_{s,SP} \rangle \langle \Gamma^{\uparrow}_{SP,i} \rangle}{\langle \Gamma^{\uparrow}_{SP} \rangle}$$

Assuming a single channel, $\langle \Gamma^{\uparrow}_{SP} \rangle = \langle \Gamma^{\uparrow}_{SP,i} \rangle$, leads to

$$\sum_s \Gamma_s(E_s) \simeq \langle \Gamma^{\downarrow}_{s,SP} \rangle \tag{4.51}$$

Hence the effect of the fine-structure resonances is to increase the width of the single-particle resonance by the sum of the fine-structure widths. This follows from the expression for the transmission factor, (4.20), which for the present application is given by

$$T = \frac{\langle \Gamma^{\downarrow}_{s,SP} \rangle \langle \Gamma^{\uparrow}_{SP,i} \rangle}{(E - \mathscr{E}_{SP})^2 + \frac{1}{4}[\langle \Gamma^{\downarrow}_{s,SP} \rangle + \langle \Gamma^{\uparrow}_{SP,i} \rangle]^2} \tag{4.52}$$

Thus a coherent hierarchical picture emerges. Because of the single-particle resonance the widths of the fine-structure resonances are enhanced [Eq. (4.50)]. Because of the doorway state resonance there is a substructure within the single-particle resonance (4.43), the enhancement of the fine-structure resonances being greater in the neighborhood of the doorway state energy. Upon averaging over the fine structure, using an averaging energy interval small compared with Γ_d but large compared to Γ_s, the doorway state broadens (the effect of $\bar{W}_{dd}$). If that broadening is too large, the doorway state resonance may not be visible. This in fact happens frequently to the single-particle resonance; that is, on averaging over an interval small compared to Γ_{SP} but large compared to Γ_d

and Γ_s, the width of the single-particle resonance becomes sufficiently broad as not to be visible in the optical model cross sections.

5. PROJECTION OPERATORS AND ANTISYMMETRIZATION

So far it has not been necessary to specify the projection operators P, Q, and so on, used in the preceding sections. Only their existence has been assumed. This fact points to the great generality of the results obtained. They apply to any system, not only the nuclear one, which exhibits both prompt and time-delayed reaction phenomena. The choice of these operators can be made so as to take into account the physics of the situation under discussion, thereby tailoring the reaction formalism to the phenomena to be understood. On the other hand, by making specific choices, one can obtain a variety of less general reaction formalisms.

The Pauli principle must, of course, be taken properly into account. The wave functions for the projectile and for the target nucleus are always assumed to be separately antisymmetrized. The exit particle and the residual nucleus wave functions are similarly treated. However, a problem arises whenever both the projectile (exit particle) and the target (residual) nucleus contain the same kinds of particles. The situation is simple if the incident particle is an electron and the process is electron elastic or inelastic scattering. In that case it is only necessary to use antisymmetric target and residual wavefunctions. But when the incident particle is a nucleon or a heavy ion, the Pauli principle must be applied to the entire system consisting of $(A_T + A_p)$ nucleons, where A_T is the target mass number and A_p that of the projectile. The Pauli principle does introduce complications, which are physically important, and therefore cannot be ignored, tempting as that may be.

The principal method to be used in this volume is chosen so that it permits a simple and simultaneous treatment of both the direct and resonance processes. As an example, consider scattering processes, both elastic and inelastic, so that the target nucleus can remain in its ground state ψ_0 or may be excited to any of a number of excited states, $\psi_1, \psi_2, \ldots, \psi_N$ with excitation energies $\varepsilon_1, \varepsilon_2, \ldots$. We consider first the case where the projectile is not composed of nucleons. Then the wavefunction of the system can be written asymptotically as

$$\Psi \to \sum_0^N u_\nu(\mathbf{r}_0)\psi_\nu(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_A) \tag{5.1}$$

where $\mathbf{r}_0$ is the coordinate of the incident particle with respect to the center of mass of the target nucleus. We shall not explicitly indicate the spin and isospin coordinates so that $\mathbf{r}_i$ contains these as well as the spatial coordinates.

The simplest prompt channel projection operator is given by

$$P\Psi = \sum_0^N u_\nu(\mathbf{r}_0)\psi_\nu(\mathbf{r}_1, \ldots, \mathbf{r}_A) \tag{5.2}$$

so that

$$P = \sum_0^N \psi_v(\mathbf{r}_1 \cdots \mathbf{r}_A) \rangle \langle \psi_v(\mathbf{r}'_1 \cdots \mathbf{r}'_A) \tag{5.3}$$

where the ψ_v's are normalized. The dependence on $\mathbf{r}_0$ and $\mathbf{r}'_0$ is not given since it is the unit operator that spatially would be $\delta(\mathbf{r}_0 - \mathbf{r}'_0)$. Comparing (5.3) and (5.2) we see that

$$u_v(\mathbf{r}_0) = \langle \psi_v(\mathbf{r}_1 \cdots \mathbf{r}_A) | \Psi(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_A) \tag{5.4}$$

Asymptotically for $v \neq 0$, $u_v \to 0(e^{ik_v r_0}/r_0)$, where $k_v = \sqrt{(2m/\hbar^2)(E - \varepsilon_v)}$ where ε_v is the excitation energy of the vth excited state of the target nucleus. When $\varepsilon_v > E$, the wave function u_v decreases exponentially with increasing r_0 for large r_0. It is not necessary to limit the sums in (5.2) and (5.3) to the open channels. If there is evidence that there are other channels of importance for the prompt processes, as might be the case for the multistep processes, these can readily included by extending the sum.

Using the P above, the equation for the prompt processes

$$(E - H_{PP})(P\Psi) = 0$$

becomes a set of coupled equations for u_v:

$$(E - H_{vv})u_v = \sum_{v' \neq v} H_{vv'} u_{v'} \tag{5.5}$$

where

$$H_{vv'} = \langle \psi_v | H \psi_{v'} \rangle \tag{5.6}$$

The effect of averaging over the fine-structure resonances is to replace H_{PP} by the optical model H so that H_{vv} is no longer real and $H_{vv'}$ no longer Hermitian. Equations (5.5) are the equations for the coupled-channel method. These are clearly appropriate if the incident projectile is an electron, a muon, or a pion, for example.

If the incident projectile is a nucleon or an ion, two additional features must be considered. The first is the Pauli principle; the second is the possibility of transfer reactions either as open channels or as an intermediate state in a multistep process. For the present we shall, for simplicity, not include transfer processes. The discussion of the projection operator needed for those cases will be deferred to Chapter VII.

Assume, then, that the incident particle is a nucleon and that asymptotically

$$\Psi \to \mathscr{A} \sum_0^N u_v(\mathbf{r}_0)\psi_v(\mathbf{r}_1, \ldots, \mathbf{r}_A) \tag{5.7}$$

where $\mathscr{A}$ is the antisymmetrization operator. It is still possible to use

$$P\Psi = \sum_{0}^{N} u_v(\mathbf{r}_0)\psi_v(\mathbf{r}_1, \ldots, \mathbf{r}_A) \tag{5.2}$$

but then the sum must be extended to highly excited states (N very large), for the following reason. Even though (5.2) is not properly antisymmetrized, it is still possible to obtain a correct result by antisymmetrizing after solving the many-body Schrödinger equation. That is, it is possible to assume that the incident nucleon is distinguishable from the nucleons in the target, solve that problem exactly, and then antisymmetrize. However, that solution must contain the open channel in which the incident nucleon and a target nucleon exchange, a process referred to as *exchange scattering*. In other words, as one lets, for example, r_1 approach infinity, Ψ will contain an outgoing wave in this variable. This can only occur if the sum, (5.2), includes the continuum components of the set of target nuclear wave functions; that is, (5.2) would need to be written

$$\sum u_v\psi_v + \int u(v)\psi(v)\,dv$$

Moreover, $u(v)$ must be a singular function of v in order that a finite amplitude exist when $r_1, r_2, \ldots$ approaches infinity.

Antisymmetrization is most essential if a finite and minimal number of coupled channels is to be used. To see how antisymmetrization helps, consider the case of elastic scattering. Let the scattering without exchange be described by $u_0(\mathbf{r}_0)\psi_0(\mathbf{r}_1, \mathbf{r}_2; \ldots, \mathbf{r}_A)$ and the exchange scattering, in which $\mathbf{r}_i$ and $\mathbf{r}_0$, are exchanged given by $v_i(\mathbf{r}_i)\psi_0(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{i-1}, \mathbf{r}_0\mathbf{r}_{i+1}, \ldots, \mathbf{r}_A)$, so that the antisymmetrized Ψ is given asymptotically by

$$\Psi \to \mathscr{A}\left[u_0(\mathbf{r}_0)\psi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) + \sum_i v_i(\mathbf{r}_i)\psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_{i-1}, \mathbf{r}_0, \mathbf{r}_{i+1}, \ldots, \mathbf{r}_A) \right]$$

Using the properties of the antisymmetrization operator this expression can be written as follows:

$$\Psi \to \mathscr{A}\left\{ \left[u_0(\mathbf{r}_0) - \sum_i v_i(\mathbf{r}_0) \right] \psi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \right\}$$

Thus the term in the antisymmetrized Ψ, which is proportional to ψ_0, contains both the direct and the exchange amplitude. Importantly, it is not necessary to include in the sum being antisymmetrized contributions from the continuum target wavefunctions to obtain the correct scattering. In other words, the amplitude of the continuum wave functions need not be singular. Of course, if either method is used, antisymmetrization after solving the Schrödinger or

antisymmetrization of the wave function before solution, the same result will be obtained. However, if approximations are to be used, particularly by truncating the system of coupled-channel equations, a better approximation will be obtained if the approximate wave function is antisymmetrized before attempting a solution.

We therefore assume that

$$P\Psi = \mathscr{A}\left[\sum_{\nu=1}^{N} u_\nu(\mathbf{r}_0)\psi_\nu(\mathbf{r}_1 \cdots \mathbf{r}_A)\right] \tag{5.8}$$

where again N may be larger than the number of open channels. Our problem will be to determine P and thereby Q from this equation.

The form of the wave function given by (5.8) is used by many authors in discussing prompt reactions. However, as pointed out by Bell (62) and Villars (77), the set $\{u_\nu\psi_\nu\}$ is overcomplete because of the antisymmetrization required by the Pauli exclusion principle. To see this, consider Fig. 5.1, which shows two possible states of the system, the incident particle being indicated by an open circle, some of the target nucleons by the filled ones. The target wave function corresponding to Fig. 5.1*a*, say ψ_a, obviously differs from that for Fig. 5.1*b*, say ψ_b, so that each would give rise to separate terms, $u_a\psi_a$ and $u_b\psi_b$, in expansion equation (5.8). However, because of the identity of the particles, there is no difference between the states of the total system illustrated in Fig. 5.1. In other words, $\mathscr{A}[u_a\psi_a]$ and $\mathscr{A}[u_b\psi_b]$ are not independent. There must be a linear combination of the two which is identically zero.

More generally, there will be functions u'_ν that satisfy the equation

$$\mathscr{A}\sum_{\nu}[u'_\nu(\mathbf{r}_0)\psi_\nu(\mathbf{r}_1 \cdots \mathbf{r}_A)] = 0 \tag{5.9}$$

The functions u'_ν satisfying (5.9) will be referred to as *superfluous*. As pointed out by the example given above, one can always add to the series in (5.8) terms coming from exchange scattering without changing $P\Psi$. To demonstrate another example of the solution of (5.9), note that if ψ_ν is a Slater determinant made up of single-particle wave functions, (5.9) is satisfied if u'_ν is any one of these

(*a*) (*b*)

FIG. 5.1

wavefunctions. The existence of solutions to (5.9) demonstrates that (5.8) does not provide a complete definition of P. Additional conditions are required.

Elimination of the superfluous solutions is essential for correct treatment of the Pauli exclusion principle. Serious errors can result if such a procedure is not carried out and (5.8) is used, assuming that all the u_ν are independent. Friedman (67) has carried out a sample calculation, the scattering of neutrons on ^{16}O, and has shown that neglect of the fact that $\{u_\nu\}$ forms a dependent set, leads to gross errors in the energies and widths of the resonances, and the prediction of spurious resonances, which are not present in the results obtained with the exact calculation (see Table 5.1).

The needed condition that will eventually permit the determination of P is suggested by the discussion preceding (5.8): namely, that the remainder of the wave function

$$Q\Psi = (1 - P)\Psi$$

does not contain any dependence on the set $\{\psi_\nu\}$ present in (5.9). This is guaranteed[‡] if

$$\langle \psi_\nu(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_A) | (1 - P)\Psi(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_A) \rangle = 0 \qquad 0 \leqslant \nu \leqslant N \tag{5.10}$$

for all $\mathbf{r}_0$. Inserting $P\Psi$ from (5.8), (5.10) becomes

$$U_\nu(\mathbf{r}_0) = u_\nu(\mathbf{r}_0) - \sum_{\nu'=0}^{N} \langle K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0) | u_{\nu'}(\mathbf{r}) \rangle \tag{5.11}$$

where

$$U_\nu(\mathbf{r}_0) \equiv \langle \psi_\nu(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_A) | \Psi(\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_A) \rangle \tag{5.12}$$

and

$$K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0) \equiv A \langle \psi_\nu(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_A) | \psi_{\nu'}(\mathbf{r}_0, \mathbf{r}_2, \ldots, \mathbf{r}_A) \rangle \tag{5.13}$$

Note[§] that K is just the one-body density matrix of (I.4.4) in deShalit and Feshbach (74) and of (I.11.11).

[‡]In (5.10) and below, integration is performed over the variables common to the bra and ket of the matrix element.

[§]Kerman (67) points out that K represents the lack of orthogonality between $\mathscr{A}(u_\nu\psi_\nu)$ and $\mathscr{A}(u_{\nu'}\psi_{\nu'})$ although ψ_ν is orthogonal to $\psi_{\nu'}$. Indeed,

$$\frac{1}{A+1} \langle \mathscr{A}u_\nu\psi_\nu | \mathscr{A}u_{\nu'}\psi_{\nu'} \rangle = \langle u_\nu\psi_\nu | \mathscr{A}u_{\nu'}\psi_{\nu'} \rangle = \langle u_\nu | (\delta_{\nu\nu'} - K_{\nu\nu'})u_{\nu'} \rangle$$

Hence orthogonality will only occur if $K_{\nu\nu'} = \delta_{\nu\nu'}$, that is, if K is diagonal with eigenvalues of unity.

TABLE 5.1[a] Scattering of Neutrons by ^{16}O

Primary Calculation		Control Calculation	
Energy E_r	Half-Width $\Gamma/2$	Energy E_r	Half-Width $\Gamma/2$
	$j = \frac{5}{2}$		
(MeV)	(keV)	(MeV)	(keV)
10.86	0.01	9.67	0.34
10.98	2.5	10.00	4.9
11.01	8.5	10.02	0.22
12.90	0.58	10.55	0.24
13.36	1.7	10.67	0.09
14.48	0.73	11.05	8.3
14.92	10	12.93	0.53
15.52	2.2	13.37	1.6
		14.50	0.83
		14.93	11
		15.52	2.1
	$j = \frac{7}{2}$		
(MeV)	(keV)	(MeV)	(keV)
5.94	120	5.24	38
6.97	57	6.72	21
7.88	7.0	7.77	0.009
9.19	42	8.01	39
10.07	1.0	9.03	35
11.21	0.12	9.80	1.1
11.47	0.10	10.16	28
12.54	22	10.92	0.002
13.79	36	11.27	0.01
15.15	15	12.57	18
		13.79	35
		15.15	15
	$j = \frac{9}{2}$		
(MeV)	(eV $\times 10^{-3}$)	(MeV)	(eV $\times 10^{-2}$)
5.53	0.53	5.92	1.3
7.33	0.03	7.15	0.36
10.58	190	9.13	0.18
10.75	1.4	10.13	0.009
14.18	54	10.00	190
		14.19	59
	$j = \frac{11}{2}$		
(MeV)	(eV)	(MeV)	(eV)
6.75	0.25	5.33	0.48
		8.97	0.54

Source: Friedman (67).

[a]Resonance parameters for the "exact" case, the primary calculation, and a case in which the projectile was given some properties which distinguished it, the control calculation thus checking the effect of the Pauli principle.

It is convenient to define the matrices

$$u \equiv \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} \qquad U \equiv \begin{pmatrix} U_0 \\ U_1 \\ \vdots \\ U_N \end{pmatrix} \qquad K = \int \begin{pmatrix} K_{00} & \cdots & K_{0N} \\ \vdots & & \vdots \\ K_{N0} & \cdots & K_{NN} \end{pmatrix} \tag{5.14}$$

so that (5.11) becomes

$$U = (1 - K)u \tag{5.15}$$

where

$$1 \equiv \int \delta(\mathbf{r} - \mathbf{r}')\delta_{\nu\nu'}$$

Note that

$$\left\langle \psi_\nu(\mathbf{r}_1 \cdots \mathbf{r}_A) \middle| \mathscr{A} \sum_{\nu'} u_{\nu'}(\mathbf{r}_0)\psi_{\nu'}(\mathbf{r}_1 \cdots \mathbf{r}_A) \right\rangle = [(1 - K)u]_\nu \tag{5.16}$$

We now consider the properties of K. We observe that K *is Hermitian*, that is,

$$K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0) = K^*_{\nu'\nu}(\mathbf{r}_0, \mathbf{r})$$

Therefore, K can be diagonalized and its eigenvalues are real:

$$K\omega_\alpha = \kappa_\alpha \omega_\alpha \qquad \langle \omega_\alpha, \omega_\beta \rangle = \delta_{\alpha\beta} \tag{5.16'}$$

Second $(1 - K)$ is positive definite. For any u [see (5.16)]

$$\langle u|(1 - K)u\rangle = \left\langle \sum_\nu u_\nu(\mathbf{r}_0)\psi_\nu(\mathbf{r}_1 \cdots \mathbf{r}_A) \middle| \mathscr{A} \left\langle \sum_{\nu'} u_{\nu'}(\mathbf{r}_0)\psi_{\nu'}(\mathbf{r}_1 \cdots \mathbf{r}_A) \right\rangle \right.$$

Since the ket is antisymmetric, one can rewrite this equation as

$$\langle u|(1 - K)u\rangle = \frac{1}{A + 1} \langle \mathscr{A} \textstyle\sum u_\nu \psi_u | \mathscr{A} \sum u_{\nu'} \psi_{\nu'} \rangle \tag{5.17a}$$

$$\geqslant 0 \tag{5.17}$$

concluding the proof.

From these results one can immediately conclude that the eigenvalues of K,

κ_α, are less than or equal to 1:

$$\kappa_\alpha \leqslant 1 \tag{5.18}$$

Problem. Show that $K_{\nu\nu}$ is positive definite. Therefore, if $P\Psi$ contains only the elastic channel, so that $K = K_{00}$,

$$1 \geqslant \kappa_\alpha \geqslant 0$$

Starting from (5.17*a*), one can derive a further bound on κ_α. Expand $\mathscr{A}\sum u_\nu\psi_\nu$ in a complete set of target functions $\psi_t(\mathbf{r}_1 \cdots \mathbf{r}_A)$. It is very important to bear in mind that it would extend beyond N, the truncation value given by (5.8). Then from (5.17*a*),

$$\begin{aligned}\langle u|(1-K)u\rangle &= \frac{1}{A+1}\sum_{t,t'=0}^{\infty}\left\langle \sum_\nu u_\nu(1-K)_{\nu t}\psi_t \middle| \sum_{\nu'} u_{\nu'}(1-K)_{\nu' t'}\psi_{t'}\right\rangle \\ &= \frac{1}{A+1}\sum_t \left\langle \sum_\nu u_\nu(1-K)_{\nu t}\psi_t \middle| \sum_{\nu'} u_{\nu' t}(1-K)_{\nu' t}\psi_t\right\rangle\end{aligned}$$

If we drop the terms for both t and $t' > N$ we drop a positive quantity, so that

$$\langle u|(1-K)u\rangle \geqslant \frac{1}{A+1}\sum_{t,t'=0}^{N}\left\langle \sum_\nu u_\nu(1-K)_{\nu t}\psi_t \middle| \sum_{\nu'} u_{\nu'}(1-K)_{\nu' t'}\psi_{t'}\right\rangle$$

Now insert for u an eigenfunction of K, ω_α, with eigenvalue κ_α. One obtains

$$(1-\kappa_\alpha) \geqslant \frac{1}{A+1}(1-\kappa_\alpha)^2$$

so that

$$\kappa_\alpha \geqslant -A \tag{5.18'}$$

which, together with (5.18), bounds κ_α between $-A$ and 1.

Problem. Prove that

$$\operatorname{tr} K = A(N+1) \tag{5.19}$$

where the trace of a matrix K with elements $K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0)$ is defined as

$$\operatorname{tr} K = \sum_\nu \int d\mathbf{r} K_{\nu\nu}(\mathbf{r}, \mathbf{r}) \tag{5.20}$$

Problem. Prove that

$$\operatorname{tr} K^2 \leqslant (\operatorname{tr} K)^2 = A^2(N+1)^2$$

The properties of K for elastic and inelastic scattering have been discussed extensively by Friedman (67).

One example of K will prove instructive. Suppose that $N = 0$; that is, the prompt channel deals only with elastic scattering. K has only one element, K_{00}:

$$K_{00} = A\langle \psi_0(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_A) | \psi_0(\mathbf{r}_0, \mathbf{r}_2, \ldots, \mathbf{r}_A \rangle \tag{5.21}$$

where ψ_0 is the ground-state wave function. Suppose that this wave function is a Slater determinant constructed of A single-particle wave functions, w_i; then it is easy to show that

$$K_{00} = \sum_1^A w_\alpha^*(\mathbf{r}) w_\alpha(\mathbf{r}_0) \tag{5.22}$$

For this K the eigenfunctions satisfying

$$K\omega_\alpha = \kappa_\alpha \omega_\alpha$$

are

$$\omega_\alpha(\mathbf{r}) = w_\alpha(\mathbf{r})$$

and

$$\kappa_\alpha = 1 \tag{5.23}$$

Note that, as should have been apparent from the discussion, the eigenfunctions of K do not form a complete set. For this case, however, $\mathscr{A}u_\nu\psi_\nu$ and $\mathscr{A}u_{\nu'}\psi_{\nu'}$ are orthogonal.

Problem. Consider a two-particle ψ_ν. Each ψ_ν is a Slater determinant made up of two single-particle wave functions taken from the set w_0, w_1, w_2: the ground state ψ_0 involving w_0 and w_1, ψ_1 just w_1 and w_2 and ψ_2, w_0 and w_2. Show that the K matrix in this example is

$K =$

$$\begin{bmatrix} w_0^*(\mathbf{r})w_0(\mathbf{r}_0) + w_1^*(\mathbf{r})w_1(\mathbf{r}_0) & \omega_0^*(\mathbf{r})w_2(\mathbf{r}_0) & w_1^*(\mathbf{r})w_2(\mathbf{r}_0) \\ w_2^*(\mathbf{r})w_0(\mathbf{r}_0) & w_1^*(\mathbf{r})w_1(\mathbf{r}_0) + w_2^*(\mathbf{r})w_2(\mathbf{r}_0) & -w_1^*(\mathbf{r})w_0(\mathbf{r}_0) \\ w_2^*(\mathbf{r})w_1(\mathbf{r}_0) & -w_0^*(\mathbf{r})w_1(\mathbf{r}_0) & w_0^*(\mathbf{r})w_0(\mathbf{r}_0) + w_2^*(\mathbf{r})w_2(\mathbf{r}_0) \end{bmatrix}$$

Show that eigenvalues of K are 1 with one exception, for which the eigenvalue is -2.

Returning to (5.15) it is obvious that any inversion of that equation must treat eigenstates of K with eigenvalues of 1 with care. The corresponding eigenfunctions are denoted $\omega_\alpha^{(1)}$. These solutions are just the *superfluous* solutions mentioned below (5.9). To prove this, consider

$$\langle \psi_\nu | \mathscr{A} \sum_{\nu'} \omega_{\alpha\nu'}^{(1)} \psi_{\nu'} \rangle = \omega_{\alpha\nu}^{(1)} - \sum_{\nu'} \langle K_{\nu\nu'} | \omega_{\alpha\nu'}^{(1)} \rangle$$

But the right-hand side equals zero because $\omega_\alpha^{(1)}$ is an eigenstate of K with eigenvalue of 1, so that

$$\left\langle \psi_\nu(\mathbf{r}_1 \cdots \mathbf{r}_A) \middle| \mathscr{A} \sum_{\nu'} \omega_{\alpha\nu'}^{(1)}(\mathbf{r}_0) \psi_{\nu'} \right\rangle = 0$$

It thus follows that

$$\left\langle \mathscr{A} \sum_{\nu} \omega_{\alpha\nu}^{(1)}(\mathbf{r}_0) \psi_\nu \middle| \mathscr{A} \sum_{\nu'} \omega_{\alpha\nu'}^{(1)}(\mathbf{r}_0) \psi_{\nu'} \right\rangle = 0$$

and therefore

$$\mathscr{A} \sum_{\nu} \omega_{\alpha\nu}^{(1)} \psi_\nu = 0 \tag{5.9}$$

the condition for the superfluous solution. Moreover, and this is an important point, the final expression for P need not contain any dependence on $\omega_\alpha^{(1)}$, since these terms will not contribute to

$$P\Psi = \mathscr{A} \sum u_\nu \psi_\nu$$

Hence the use of the term *superfluous*. This fact manifests itself in a property of U defined by (5.12) and (5.14), namely

$$\langle \omega_\alpha^{(1)} | U \rangle = \sum \langle \omega_{\alpha\nu}^{(1)} | U_\nu \rangle = 0 \tag{5.24}$$

To prove this, insert (5.15) for U, yielding

$$\langle \omega_\alpha^{(1)} | U \rangle = \langle \omega_\alpha^{(1)} | (1 - K) u \rangle$$

Because of the hermiticity of K,

$$\langle \omega_\alpha^{(1)} | U \rangle = \langle (1 - K) \omega_\alpha^{(1)} | U \rangle = 0$$

Thus the superfluous solutions are orthogonal to U.

We now return to (5.15):

$$U = (1 - K)u$$

The general solution of this equation is

$$u = \sum_{\kappa_\alpha \neq 1} \omega_\alpha \frac{\langle \omega_\alpha | U \rangle}{1 - \kappa_\alpha} + \text{terms in } \omega_\alpha^{(1)}$$

In accordance with the discussion just concluded, the terms in $\omega_\alpha^{(1)}$ are dropped, so that

$$\langle \omega_\alpha^{(1)} | u \rangle = 0 \tag{5.25}$$

and

$$u = \sum_{\kappa_\alpha \neq 1} \omega_\alpha \frac{\langle \omega_\alpha | U \rangle}{1 - \kappa_\alpha} \equiv \frac{1}{1 - K'} U \tag{5.26}$$

where the operator $1/1 - K'$ is defined by

$$\frac{1}{1 - K'} = \sum_{\kappa_\alpha \neq 1} \frac{\omega_\alpha \rangle \langle \omega_\alpha}{1 - \kappa_\alpha} \tag{5.27}$$

It is convenient to rewrite (5.26) as follows:

$$u = U + \sum_{\kappa_\alpha \neq 1} \omega_\alpha \langle \omega_\alpha | U \rangle \left(\frac{1}{1 - \kappa_\alpha} - 1 \right)$$

where we have made use of (5.24). Finally,

$$u = U + \sum_{\kappa_\alpha \neq 1} \omega_\alpha \langle \omega_\alpha | U \rangle \frac{\kappa_\alpha}{1 - \kappa_\alpha} \tag{5.28}$$

In terms of K',

$$u = \frac{1}{1 - K'} U = U + \frac{K'}{1 - K'} U \tag{5.29}$$

Recalling that $U_\nu \equiv \langle \psi_\nu | \Psi \rangle$, one obtains

$$P\Psi = \mathscr{A} \sum u_\nu \psi_\nu = \mathscr{A} \sum_\nu \left[U_\nu + \sum_{\kappa_\alpha \neq 1} \omega_{\alpha\nu} \langle \omega_\alpha | U \rangle \frac{\kappa_\alpha}{1 - \kappa_\alpha} \right] \psi_\nu$$

In terms of K',

$$P = \mathscr{A}\left[\sum_r \psi_\nu\right\rangle\left\langle\psi_\nu + \sum_{\nu,\nu'}\psi_\nu\right\rangle\left(\frac{K'}{1-K'}\right)_{\nu\nu'}\langle\psi_{\nu'}\Big] = \mathscr{A}\left[\sum_{\nu\nu'}\psi_\nu\right\rangle\left(\frac{1}{1-K'}\right)_{\nu\nu'}\langle\psi_{\nu'}\Big] \tag{5.30}$$

This is the projection operator which, when applied to Ψ, yields $\mathscr{A}\sum u_\nu\psi_\nu$, with the requirement that none of the ψ_ν which appear in this sum are present in $Q\Psi$. This requirement must be satisfied if $P\Psi$ is to involve a finite number of ψ_ν. Equation (5.30) reduces to the simple form $\sum_\nu \psi_\nu\rangle\langle\psi_\nu$ when all the κ_α are either zero or 1.

With P determined, it now becomes possible to develop an explicit statement for the Schrödinger equation (2.3) and (2.4) or (2.7). Let us use the last form,

$$[E - (H_{\text{eff}})_{PP}]P\Psi = 0 \tag{5.31}$$

To reduce this equation to a set of coupled equations in one variable, namely the distance between the projectile and the center of mass of the target, premultiply (5.31) successively by ψ_ν and integrate over all the coordinates of the target nucleons. Note from (5.10) and (5.12) that

$$U_\nu = \langle\psi_\nu|\Psi\rangle = \langle\psi_\nu|P\Psi\rangle \tag{5.32}$$

Since Ψ is an arbitrary antisymmetric function of $(A+1)$ variables, the second pair of equations holds for any such function. Equation (5.31) becomes the set of coupled equations

$$EU_\nu = \sum_{\nu',\nu''}\left\langle\psi_\nu\middle|H_{\text{eff}}\mathscr{A}\sum_{\nu',\nu''}\psi_{\nu'}\left(\frac{1}{1-K'}\right)_{\nu'\nu''}U_{\nu''}\right\rangle \tag{5.33}$$

In this equation, the effect of the projection appears explicitly in the Hamiltonian term. If we now introduce the function u_ν, (5.33) becomes

$$E\sum_{\nu'}(1-K')_{\nu\nu'}u_{\nu'} = \left\langle\psi_\nu\middle|H_{\text{eff}}\mathscr{A}\sum_{\nu'}\psi_{\nu'}u_{\nu'}\right\rangle \tag{5.34}$$

The term on the right involves both direct and exchange terms. The term on the left is modified by the inclusion of normalization corrections. Note, however, the explicit omission of the eigenfunctions of K with unit eigenvalue. Since $K'_{\nu\nu'}$ can be written as a finite sum of factorable terms its inclusion does not present any special difficulties. If all the eigenvalues of K are 1 or zero, $K' = 0$, with the result that one need only include the exchange terms in the effective Hamiltonian to take account of antisymmetrization.

An alternative form for (5.34) is obtained by reintroducing the operator K:

$$E\sum_{\nu'}(1-K_{\nu\nu'})u_{\nu'}+E\sum_{\alpha\nu'}\omega_{\alpha\nu}^{(1)}\rangle\langle\omega_{\alpha\nu'}^{(1)}|u_{\nu'}\rangle=\left\langle\psi_\nu\middle|H_{\text{eff}}\mathscr{A}\sum_{\nu'}u_{\nu'}\psi_{\nu'}\right\rangle \tag{5.35}$$

Or because of the orthogonality between $\omega_\alpha^{(1)}$ and u,

$$E\left\langle\psi_\nu\middle|\mathscr{A}\sum_{\nu'}u_{\nu'}\psi_{\nu'}\right\rangle=\left\langle\psi_\nu\middle|\mathscr{H}_{\text{eff}}\mathscr{A}\sum_{\nu'}u_{\nu'}\psi_{\nu'}\right\rangle$$

$$\langle\omega_\alpha^{(1)}|u\rangle=0 \tag{5.36}$$

In other words, one can use the naive wave function $\mathscr{A}\sum u_\nu\psi_\nu$ in the Schrödinger equation as long as u is orthogonal to all the eigenstates $\omega_\alpha^{(1)}$ of K with eigenvalue 1. Which form, (5.34) or (5.36), is used is a matter of convenience. If the $\omega_\alpha^{(1)}$ are known a priori, it might be more convenient to use (5.36). On the other hand, if all the eigenvalues of K are zero or 1, (5.34) might prove more useful, especially since the orthogonality $\omega_\alpha^{(1)}$ to u is an automatic consequence of (5.34).

Orthogonality of $\omega_\alpha^{(1)}$ can be enforced by projection:

$$\bar{u}=u-\sum_\alpha\omega_\alpha^{(1)}\langle\omega_\alpha^{(1)}|u\rangle$$

The resulting $\mathscr{A}\sum\bar{u}_\nu\psi_\nu$ remains a solution of (5.36) since

$$\mathscr{A}\sum_\nu\omega_{\alpha\nu}^{(1)}\psi_\nu$$

is identically zero for all α [Saito (69)].

Another method that is particularly useful when asymptotic conditions need to be satisfied explicitly is used by Auerbach, Gal et al (72). For example, rewrite (5.34) [or (5.35)] making the orthogonality of $\omega_\alpha^{(1)}$ explicit by introducing the projection operator q_1 and p_1

$$q_1\equiv 1-\sum_\alpha\omega_\alpha^{(1)}\rangle\langle\omega_\alpha^{(1)}=1-p_1 \tag{5.37}$$

Then both equations can be written in the form

$$q_1E\rho u=q_1hu \tag{5.38}$$

If (5.34) is used, ρ is just $1-K'$. The identical result is obtained if (5.35) is used.

Writing $q_1=1-p_1$, (5.38) becomes

$$(E\rho-h)u=p_1(E\rho-h)u$$

Let

$$(E\rho - h)u^{(0)} = 0$$

and

$$(E\rho - h)\mathscr{G}^{(+)} = 1$$

Then

$$\begin{aligned} u &= u^{(0)} + \mathscr{G}^{(+)}p_1(E\rho - h)u \\ &= u^{(0)} + \sum \mathscr{G}^{(+)}\omega_\alpha^{(1)}\rangle\langle\omega_\alpha^{(1)}|(E\rho - h)u\rangle \end{aligned}$$

The orthogonality requirement becomes

$$\langle\omega_\beta^{(1)}|u\rangle = \langle\omega_\beta^{(1)}|u^{(0)}\rangle + \sum_\alpha \langle\omega_\beta^{(1)}|\mathscr{G}^{(+)}\omega_\alpha^{(1)}\rangle\langle\omega_\alpha^{(1)}|(E\rho - h)u\rangle = 0$$

Defining the matrix $\hat{g}$ by

$$g_{\beta\alpha} = \langle\omega_\beta^{(1)}|\mathscr{G}^{(+)}\omega_\alpha^{(1)}\rangle$$

these equations can be solved for $\langle\omega_\alpha^{(1)}|(E\rho - h)u\rangle$:

$$\langle\omega_\alpha^{(1)}|(E\rho - h)u\rangle = -\sum_p (\hat{g}^{-1})_{\alpha\beta}\langle\omega_\beta^{(1)}|u^{(0)}\rangle$$

Finally,

$$u = u^{(0)} - \sum_\alpha \mathscr{G}^{(+)}\omega_\alpha^{(1)}\rangle(\hat{g}^{-1})_{\alpha\beta}\langle\omega_\beta^{(1)}|u_0\rangle \tag{5.39}$$

Problem. In (5.34) introduce the wave functions

$$\Omega_\nu = \sum_{\nu'} (\sqrt{1 - K'})_{\nu\nu'}u_{\nu'}$$

and derive the coupled equations for Ω_ν. Show that the effective Hamiltonian for this equation is Hermitian.

It is perhaps unnecessary to add that H_{eff} in (5.34) or (5.36) can be replaced by H_{opt} and thus provide a description of the prompt processes after averaging over the fine structure. The supplementary condition in (5.36) must, of course, still be satisfied.

In summary, to take into account the Pauli exclusion principle and simultaneously limit the space $\mathscr{P}$, one first evaluates K. This matrix depends only on the ground state, ψ_0, of the target nucleus and a finite number of its

excited states, ψ_ν. Once K is obtained, and its eigenstates ω_α determined, one can proceed to solve (5.34) using only those ω_α whose eigenvalues differ from zero or 1. Or using only those with eigenvalues of unity, one can solve (5.36), including the supplementary condition. Determining the eigenstates of K is not a formidable task, as K is bounded and Hermitian and depends on one coordinate. However, the target wave functions should include center-of-mass effects, which does introduce some complication.

If determinantal wave functions are used for ψ_ν, one avoids calculating K altogether, since then it is known that all the eigenvalues of K are 1 and that the eigenfunctions of K are just the single-particle wave functions making up the determinant. However, the center-of-mass problem remains.

The discussion above is restricted to scattering, elastic and inelastic. The effects of the Pauli principle for particle transfer reactions are discussed in later chapters.

6. ALTERNATIVE REACTION FORMALISMS

Such a wide variety of reaction formalisms have been proposed that it is not possible to present an adequate review of each in this volume. We have, therefore, chosen to describe a few of the more familiar examples. To the extent that each one is exact, they should give identical answers to a specific problem. The advantages of a given approach depend more on the ease with which it can be applied to a given physical situation. One should especially ask: Is it in a form that permits a direct physical interpretation of the experimental results, and similarly, does it have predictive capability? Can it make good use of what is known about the nuclear structure of the colliding systems, of the compound system, and of the produced particles? Can its parameters, in principle, if not immediately, be derived from fundamental nuclear theory, that is from the underlying nucleon–nucleon force, or less ambitiously from a model Hamiltonian plus a residual interaction?

Fortunately for the comparison with experimental data, the *form* of the results for the transition amplitudes and cross sections is the same for all these formalisms. This might be referred to as a *formalism invariance.* An important reason for this similarity lies in the requirements of unitarity. Unitarity, when applied to a *single-channel* process, requires that the S matrix satisfy

$$|S|^2 = 1 \quad \text{or} \quad S = e^{2i\delta}, \quad \delta \text{ real} \tag{6.1}$$

This condition is satisfied by the following form [see (2.44)]:

$$S = e^{2i\phi_P} \frac{1 - i\pi\mathscr{K}}{1 + i\pi\mathscr{K}} \tag{6.2}$$

where ϕ_P is a smoothly varying function of the energy and $\mathscr{K}$ is a real function

which contains that part of S which varies rapidly with energy. The function ϕ_P gives the potential scattering phase shift so that the potential scattering S matrix, S_P, is

$$S_P = e^{2i\phi_P}$$

and (6.2) can be written

$$S = S_P S_R \tag{6.3}$$

where

$$S_R = \frac{1 - i\pi\mathscr{K}}{1 + i\pi\mathscr{K}} \tag{6.4}$$

The words *smoothly* and *rapidly* are not quantitatively defined, so that the factorization equation (6.3) is not completely specified, and indeed, different reaction theories will use differing S_P.

The transition matrix $\mathscr{T}[=(1-S)/2\pi i]$ is

$$\mathscr{T} = \mathscr{T}_P + S_P \mathscr{T}_R \tag{6.5}$$

where

$$\mathscr{T}_P = \frac{1 - S_P}{2\pi i} = -\frac{1}{\pi} e^{i\phi_P} \sin\phi_P$$

and

$$\mathscr{T}_R = \frac{\mathscr{K}}{1 + i\pi\mathscr{K}} \tag{6.6}$$

Equation (6.5) has the same form as (2.47) in the single-channel context. Moreover, the resonance representation of $\mathscr{K}$ can in general be given by

$$\mathscr{K} = \frac{1}{2\pi} \sum \frac{\gamma_s^2}{E - e_s} \tag{6.7}$$

where γ_s^2 and e_s are real. In the case of a single isolated resonance, at say e_0, $\mathscr{T}$ becomes

$$\mathscr{T} = -\frac{1}{\pi} e^{i\phi_P} \sin\phi_P + e^{2i\phi_P} \frac{(1/2\pi)[\gamma_0^2/(E - e_0)]}{1 + (i/2)[\gamma_0^2/(E - e_0)]}$$

or

$$\mathscr{T} = -\frac{1}{\pi} e^{i\phi_P} \left[\sin\phi_P - e^{i\phi_P} \frac{\gamma_0^2/2}{E - e_0 + (i/2)\gamma_0^2} \right] \tag{6.8}$$

which is just the single-channel resonance formula, showing both a potential scattering term and a resonant term. Equation (6.8) is a consequence of unitarity plus the statement that the two kinds of amplitudes are involved: When there are many terms in the series for $\mathscr{K}, \mathscr{T}$ is not as simple. However, it should be possible to expand $\mathscr{T}$ in terms of the poles of the S matrix, E_μ, such as that given by (2.33):

$$\mathscr{T} = \mathscr{T}_P + e^{2i\phi_P} \sum_\mu \frac{\mathscr{A}_\mu}{E - E_\mu} \tag{6.9}$$

so that

$$\mathscr{T}_R = \sum_{\mu=1}^{n} \frac{\mathscr{A}_\mu}{E - E_\mu} = \frac{(1/2\pi)\sum_1^n [\gamma_s^2/(E - e_s)]}{1 + (i/2)\sum_1^n [\gamma_s^2/(E - e_s)]} \tag{6.10}$$

Recall that both $\mathscr{A}_\mu$ and E_μ are complex numbers, so that the number of parameters in the middle expression is $4n$ if the number on the right side is $2n$, implying that there are $2n$ relations between $\{\mathscr{A}_\mu\}$ and $\{E_\mu\}$. These relations can be obtained from (6.10). First expand both sides of (6.10) in a power series in inverse powers of E. The left-hand side gives

$$\mathscr{T}_R = \frac{1}{E}\sum \mathscr{A}_\mu + \frac{1}{E^2}\sum \mathscr{A}_\mu E_\mu + \cdots$$

and the right-hand side gives

$$\mathscr{T}_R = \frac{1}{2\pi}\left[\frac{1}{E}\sum \gamma_s^2 + \frac{1}{E^2}\left\{\sum \gamma_s^2 e_s - \frac{i}{2}(\sum \gamma_s^2)^2\right\} + \cdots\right]$$

so that

$$\sum \mathscr{A}_\mu = \frac{1}{2\pi}\sum \gamma_s^2$$

$$\sum \mathscr{A}_\mu E_\mu = \frac{1}{2\pi}\left[\sum \gamma_s^2 e_s - \frac{i}{2}(\sum \gamma_s^2)^2\right]$$

Since $\{\gamma_s^2\}$ is real,

$$\sum \operatorname{Im} \mathscr{A}_\mu = 0 \tag{6.11}$$

$$\sum \operatorname{Re} \mathscr{A}_\mu = \frac{1}{2\pi}\sum \gamma_s^2 \tag{6.12}$$

An alternative but not independent requirement states that the poles of the

right-hand side of (6.10) occur at E_μ. These poles are the roots of

$$\prod_{s=1}^{n}(E-e_s)+\frac{i}{2}\sum\gamma_s^2\prod_{t\neq s}(E-e_t)=0$$

The sum of the roots equals the coefficient of $-E^{(n-1)}$, so that

$$\sum E_\mu=\sum\left(e_s-\frac{i}{2}\gamma_s^2\right)$$

implying that

$$\mathrm{Re}\sum E_\mu=\sum e_s \tag{6.13}$$

and

$$\mathrm{Im}\sum E_\mu=-\tfrac{1}{2}\sum\gamma_s^2=-\pi\sum\mathrm{Re}\,\mathscr{A}_\mu \tag{6.14}$$

One can verify from (6.8) that (6.11) and (6.14) hold for an isolated resonance:

$$\mathrm{Im}\,\mathscr{A}_\mu=0$$

$$\mathrm{Re}\,\mathscr{A}_\mu=-\frac{1}{\pi}\mathrm{Im}\,E_\mu$$

Equations (6.11) and (6.14) state that these equations hold on the average for overlapping resonances. One can readily continue this process. The next order yields

$$2\pi\sum_{\mu>\nu}(\mathrm{Re}\,\mathscr{A}_\mu)(\mathrm{Re}\,E_\nu)=\mathrm{Im}\sum_{\mu>\nu}E_\mu E_\nu$$

However, these additional relationships are not informative. The major point, to be gained from the discussion following (6.10), is that the parameters $\mathscr{A}_\mu$ and E_μ are not independent, if S is to be unitary.

The sequence of equations (6.5), (6.6), (6.7), and (6.9) present an expansion of $\mathscr{T}_R$ in terms of the poles of the reactance matrix $\mathscr{K}$, which automatically satisfies unitarity or an expansion in poles of the S matrix, where, however, unitarity is not obvious but is secured through relations such as (6.11) and (6.14). The form of these results is independent of a particular reaction formalism. The difference among the formalisms lies in their statement regarding the potential scattering amplitude $\mathscr{T}_P$ and the consequent differences in the interpretation of the parameters γ_μ^2, e_μ, $\mathscr{A}_\mu$, and E_μ.

As an example, consider the potential scattering amplitude $\mathscr{T}_P$. In an early version of the $\mathscr{R}$-matrix theory (to be described below), $\mathscr{T}_P$ is taken to be hard-sphere scattering for which the phase shift for the zero angular momentum $l=0$ partial wave (and for large energies for all l) is $-kR$, where R is the radius

of the sphere. In the formalism of Section 2 it is the scattering caused by the Hamiltonian $\bar{H}_{PP}$. How can these give the same total amplitude? Clearly, hard-sphere scattering cannot be correct since it presumes an infinite potential energy. It must therefore be compensated by taking many terms in the series for $\mathscr{K}$. Summing up the effect of distant resonances, whose energy dependence over a small interval in energy is weak, modifies the hard-sphere shift, presumably into one whose energy dependence is more in accord with scattering caused by a potential. Moreover, we see that these distant resonances are not physically meaningful. Choosing the potential scattering does affect the values for the widths [see, e.g., (2.22)] and the resonance energies. Consistency requires that the same description of potential scattering is used in calculating or developing the energy dependence of the width and resonance energy.

The analysis of experimental data exhibits a similar problem. In fitting a cross section to either (6.6) and (6.7) or to (6.9), it is necessary to determine the parameters ϕ_P as well as say γ_μ and e_μ, and to decide how many terms in the series over μ to use. Changing that number will modify the empirical values of γ_μ, e_μ, and ϕ_P. The pragmatic response to this issue is to include the number of terms as one of the parameters in obtaining a fit. In other words, one looks for a number of terms such that the inclusion of an additional one in the sum over μ does not affect the χ^2 for the fit and does not modify the values of the parameters obtained from the data. One must also require that the potential phase shift does not vary rapidly with energy. It is clear that in presenting an analysis of data, the method used in obtaining the parameters should be carefully stated. Of course, these problems do not arise if the resonance is isolated.

Problem. Write the S matrix in the presence of doorway states is

$$S = S_P S_D S_r$$

where S_P is the S matrix for potential scattering, S_D for doorway state resonances, and S_r for fine-structure resonances. Show that

$$\mathscr{T} = \mathscr{T}_P + S_P \mathscr{T}_D + S_P S_D \mathscr{T}_r \tag{6.15}$$

We turn now to some examples of reaction formalisms. Our general aim will be to relate them to the general discussion given in Sections 2 to 6 of this chapter.

A. The Theory of Kapur and Peierls[‡]

This theory sets a boundary condition at each channel radius R_c. In the single-channel case, the boundary condition to be satisfied by the resonant

[‡]Kapur and Peierls (38).

state is

$$\frac{\partial X_\mu}{\partial r} = ikX_\mu \qquad \text{at } r = R \tag{6.16}$$

The radius R is any distance beyond which the nuclear interactions vanish. The consequence is a resonance series for the S matrix [see (2.46)] of the form given by (2.33). The parameters, the widths and resonance energies, are real but energy dependent. Equation (6.16) simply states that X_μ at R is an outgoing wave with wave number k. The generalization of (6.16) to several channels requires that

$$\frac{\partial X_{\mu c}}{\partial r} = ik_c X_{\mu c} \qquad \text{at } r = R_c \tag{6.17}$$

where the subscript c denotes the channel and k_c is the corresponding wave number.

The theory described in Section 2 of this chapter provides a generalization of the theory of Kapur and Peierls free of the use of a boundary condition radius or separation into partial waves. To demonstrate this point, consider the eigenfunctions Ω_μ of the operator $H_{QQ} + W_{QQ}$ discussed in the material following (2.31). They satisfy

$$(E_\mu - H_{QQ} - W_{QQ})\Omega_\mu = 0 \tag{6.18}$$

where

$$W_{QQ} = H_{QP} \frac{1}{E^{(+)} - H_{PP}} H_{PQ}$$

Note the parametric dependence of Ω_μ and E_μ on E, the energy of the system. Equation (6.18) is equivalent to the coupled equations

$$(E - H_{PP})\chi_\mu = H_{PQ}\Omega_\mu \tag{6.19a}$$

$$(E - H_{QQ})\Omega_\mu = H_{QP}\chi_\mu \tag{6.19b}$$

where χ_μ are the open-channel wave functions associated with the resonant state Ω_μ. Equation (6.18) is obtained if the solution of (6.19a) is taken to be

$$\chi_\mu = \frac{1}{E^{(+)} - H_{PP}} H_{PQ}\Omega_\mu \tag{6.19c}$$

The χ_μ, then, satisfy outgoing wave boundary conditions which are the natural generalization of (6.17), concluding the argument. As in the case of the Kapur–Peierls theory, the resonance widths and energies are functions of E.

This dependence is weak; its strength determined by the energy variation of the open-channel wave functions, that is, of the prompt amplitudes. Indeed, this energy dependence is desirable. It takes into account the effect of the energy dependence of the prompt amplitudes upon the resonance parameters. This dependence could be significant if the resonance is wide, or if many resonances in a substantial energy domain, ΔE, are being considered.

B. $\mathscr{R}$-Matrix Formalism[‡]

This method can be considered to be a special case of the formalism, described in this chapter, in which the projection operator P is geometric. Conceptually, the simplest operator, which projects out of the exact solution Ψ, a part that has the same asymptotic behavior as Ψ, is one that is unity outside the region in which the nuclear interaction takes place and zero inside; that is, $P = 1$ as long as $|\mathbf{R}_a - \mathbf{r}_a| > R_c$, where $\mathbf{r}_a$ is the coordinate of the ath nucleon, $\mathbf{R}_a$ the center of mass of the rest of the system, and R_c is the interaction radius appropriate for a channel c. The operator Q then projects into the interaction region. The problems raised by the Pauli principle are thus encountered only in $\mathscr{Q}$ space. In $\mathscr{P}$ space the form used in (5.8) suffices. This assumes that the problems raised by the long-range Coulomb forces are not important.

However, the use of a spatially discontinuous projection operator does give rise to problems that must be carefully treated. It is necessary, for example, to ensure the continuity of total Ψ; that is, the two discontinuous functions $P\Psi$ and $Q\Psi$ must join smoothly at the surface of the interaction region, $|\mathbf{R}_a - \mathbf{r}_a| = R_c$ for all a. The kinetic energy operators must be suitably defined. Since $P\Psi$ is discontinuous, $\nabla_a^2(P\Psi)$ will be singular on the surface of the interaction. Similar remarks apply to $Q\Psi$. But $\Psi = P\Psi + Q\Psi$ will have no singularities at this surface. Identical problems of this kind occur in the theory of boundary perturbations and Green's functions [Morse and Feshbach (53)] and can be resolved in the same way. The value on the surface of the interaction region of $\nabla_a^2(P\Psi)$ is taken to be its value as one approaches the surface from the field-free region, that is, as $|\mathbf{R}_a - \mathbf{r}_a| \to R_c^{(+)}$. Similarly, $\nabla_a^2(Q\Psi)$ at this surface is defined in the limit $|\mathbf{R}_a - \mathbf{r}_a| \to R_c^{(-)}$, that is, as the surface is approached from the interaction region. Continuity for Ψ is ensured, as in the Green's function case, through the use of a singular surface interaction. In the case of the Green's function for the Laplace equation, this procedure is equivalent to assuming the presence of compensating monopole and dipole layers. In the present context, we introduce a surface interaction, $\mathscr{B}$, which is present in both the field-free and interaction regions. $\mathscr{B}$ is defined by

$$\mathscr{B} = \frac{1}{A+1}\sum_\alpha \mathscr{B}_\alpha, \qquad \mathscr{B}_\alpha = b_c \delta(R_c - \rho_\alpha)\left(B_c + \frac{\partial}{\partial \rho_\alpha}\right) \tag{6.20}$$

[‡]Wigner and Eisenbud (47).

where

$$\rho_\alpha \equiv |\mathbf{R}_\alpha - \mathbf{r}_\alpha|$$

and $\partial/\partial\rho_\alpha$ is the normal derivative to the surface, $\rho_\alpha = R_c$. B_c is an arbitrary constant. The constant b_c can be determined from the single-channel case to be $\hbar^2/2\mu_c$, where μ_c is the reduced mass for the channel. The interaction region has been assumed to be spherical. It is a simple matter to generalize (6.20) to the case of a deformed shape. The coupled equations (2.3) and (2.4) are replaced by

$$(E - H_{PP} - \mathscr{B})P\Psi = -\mathscr{B}(Q\Psi) \tag{6.21}$$

$$(E - H_{QQ} - \mathscr{B})Q\Psi = -\mathscr{B}(P\Psi) \tag{6.22}$$

We see that the requirements of continuity provide the coupling between $P\Psi$ and $Q\Psi$. Continuity is assured by these equations since the singularities on both sides must match. Note that by defining $\nabla_a^2(P\Psi)$ [and $\nabla_a^2(Q\Psi)$] as indicated above, H_{PP} and H_{QQ} will not be singular at $\rho_a = R$.

The functions Φ_s of (2.9) are solutions of the homogeneous form of (6.22). We shall call these functions X_λ. They satisfy

$$(E_\lambda - H_{QQ} - \mathscr{B})X_\lambda = 0 \tag{6.23}$$

Integrating over a small interval in ρ_α containing R, we find that X_λ satisfies the boundary condition

$$\frac{\partial X_\lambda}{\partial \rho_\alpha} + B_c X_\lambda = 0 \qquad \text{at } \rho_\alpha = R_c \tag{6.24}$$

This condition, together with

$$(E_\lambda - H_{QQ})X_\lambda = 0 \tag{6.25}$$

are equivalent to (6.23). Since we are dealing with an "interior problem," the spectrum of E_λ will be discrete; moreover, because of the nature of H_{QQ}, it will be unbounded from above. The X_λ form an orthonormal complete set. The potential scattering wave functions $\chi^{(\pm)}$ are, similarly, solutions of

$$(E - H_{PP})\chi^{(\pm)} = 0 \tag{6.25'}$$

In addition to satisfying the boundary conditions at infinity indicated by their superscripts, they must join continuously with the internal wave functions, H_{PP} consists only of the kinetic energy operator and the long-range electromagnetic interactions.

These results are already sufficient to establish the connection with the

Wigner–Eisenbud $\mathscr{R}$-matrix theory. However, we can go one step further and derive the relation between the wave function and its derivative needed to obtain the transition amplitude in the $\mathscr{R}$-matrix theory. By eliminating $Q\Psi$ from (6.22), we obtain the familiar equation for $P\Psi$:

$$\left[E - H_{PP} - \mathscr{B} - \mathscr{B}\frac{1}{E - H_{QQ} - \mathscr{B}}\mathscr{B}\right]P\Psi = 0 \tag{6.26}$$

By requiring that the coefficient of the δ function singularity in the $\mathscr{B}$-dependent potential be zero, then integrating over a small interval in ρ_α including R, yields the result, after expansion in eigenfunctions X_λ, that

$$\Psi(R;\alpha) = \sum \frac{X_\lambda(R;\alpha)\langle X_\lambda|\mathscr{B}\Psi\rangle}{E_\lambda - E} \tag{6.27}$$

where $\Psi(R;\alpha)$ is the wave function Ψ evaluated with the αth particle on the interaction surface, that is, with $\rho_\alpha = R$. Because of the symmetry of X_λ and Ψ,

$$\langle X_\lambda|\mathscr{B}\Psi\rangle = \left\langle X_\lambda\middle|\delta(R-\rho_\alpha)\left(B - \frac{\partial}{\partial\rho_\alpha}\right)\Psi\right\rangle \tag{6.28}$$

To obtain the Wigner–Eisenbud result it is necessary to introduce a complete and orthonormal set of *surface* wave functions $\Xi_c^{(\alpha)}(1,2,\ldots,\alpha-1,\ \alpha+1,\ldots,A+1;\ R\hat{\Omega}_\alpha)$. The superscript α indicates that the αth particle is on the interaction surface; the remaining coordinates are for the other particles that are located within the interaction region. The subscript c indicates the channel. The orthonormality condition is

$$\int \Xi_c^{(\alpha)*}\Xi_{c'}^{(\alpha)}dS_\alpha = \delta_{cc'} \tag{6.29}$$

where the integration is over the surface $\rho_\alpha = R$ and over the interior volume for the remaining particles. Expanding $\Psi(R;\alpha)$ in terms of these wave functions yields

$$a_c = \sum_{c',\lambda}\frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}(Ba_{c'} + b_{c'}) \tag{6.30}$$

where

$$a_c = \langle \Xi_c^{(\alpha)}|\Psi\rangle$$

and

$$b_c = \left\langle \Xi_c^{(\alpha)}\middle|\frac{\partial\Psi}{\partial\rho_\alpha}\right\rangle \tag{6.31}$$

are independent of α because of the symmetry of the wave functions. The subscript on B has been dropped to avoid confusion. The quantities $\gamma_{\lambda c}$ are

$$\gamma_{\lambda c} = \langle X_\lambda | \Xi_c^{(\alpha)} \rangle \tag{6.32}$$

Equation (6.30) is the primary $\mathscr{R}$-matrix result as given by Teichmann and Wigner (52). (There are some differences in notation.) The $\mathscr{R}$ matrix is given by

$$\mathscr{R}_{cc'} = \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E} \tag{6.33}$$

The $\gamma_{\lambda c}$ are *independent* of the energy so that the energy dependence of $\mathscr{R}$ is completely explicit. We see that when $B = 0$, roughly speaking, $\mathscr{R}$ gives the relation between the magnitude of Ψ on the surface and its radial derivative.

We shall not carry this development any further. To obtain the reaction amplitudes, we need only note that the external wave functions can also be written in terms of the surface wave functions multiplied by radial wave functions which need to be adjusted so as to satisfy the joining conditions at the channel radius $\rho_\alpha = R$ given by (6.30) and the usual combination of an incident wave and an outgoing wave at $\rho_\alpha \to \infty$.

This formalism contains two arbitrary parameters, R and B, for each channel. The predictions are independent of their value. But the question can be asked if there is a best value of each, so that accurate approximations can be readily made. For example, it would be desirable if the series, (6.33), were limited to a few terms. However, there are also some important requirements that must be met which act to increase the number of terms. It would at first seem reasonable to propose the nuclear radius for R. But the intrinsic nuclear wave functions do not fall precipitously to zero at R, nor is the range of nuclear forces equal to zero. Therefore, the region beyond R is in fact not nuclear force free. One could attempt to take this into account by including these terms in H_{PP}, but then the effects of the Pauli principle would have to be explicitly considered. An alternative procedure would be to take a larger value of R, that is, one larger than the nuclear radius. But then the density of the levels E_λ increases and the number of terms in the series for $R_{cc'}$ which would need to be considered, increases correspondingly. The method adopted limits the series to a few terms in the region of energy of interest and considers the remaining terms as an empirical parameter with a weak energy dependence. The effect of this procedure is to modify the potential scattering from that given by (6.25′), in other words, to introduce a potential term in the external region. For further discussions of this problem, the reader is referred to Teichmann and Wigner (52), Thomas (55), and Lane and Thomas (50).

The use of the boundary condition operator was introduced into the theory of nuclear reactions by Bloch (57) and elaborated by Lane and Robson (66, 67). The description above is taken from Feshbach (62). It permits the direct use of

the formalism in Sections 2 to 5 to evaluate the consequence of the $\mathscr{R}$-matrix assumptions.

C. S-Matrix Formalism[‡]

The Kapur–Peierls expansion employs the eigenvalues of the operator $H_{QQ} + W_{QQ}$, where the dependence on the energy E of W_{QQ} is regarded as parametric, so that the eigenvalue E_μ and eigen function Ω_μ are both functions of E:

$$[E_\mu(E) - H_{QQ} - W_{QQ}(E)]\Omega(E) = 0 \tag{6.34}$$

This is an excellent approximation as long as $W_{QQ}(E)$ varies slowly with E. If we have been careful in selecting the prompt channels forming $\mathscr{P}$, this will be generally true unless the resonance width is an appreciable fraction of the single-particle width. However, this condition is not always satisfied, particularly when the target nucleus is light. It therefore becomes necessary to improve on this approximation by solving

$$[\mathscr{E}_\mu - H_{QQ} - W_{QQ}(\mathscr{E}_\mu)]\Lambda_\mu = 0 \tag{6.35}$$

This is the procedure that was adopted by Lemmer and Shakin (64). From Λ_μ [compare with (6.19c)] one can obtain the prompt wave function:

$$(P\Psi)_\mu = \frac{1}{\mathscr{E}_\mu^{(+)} - H_{PP}} H_{PQ}\Lambda_\mu \tag{6.36}$$

For a given channel α, this wave function satisfies the boundary condition

$$\frac{\partial(P\Psi)_\mu}{\partial\rho_\alpha} \to ik_\alpha(P\Psi)_\mu \tag{6.37}$$

where

$$k_\alpha^2 = \frac{2\mu}{\hbar^2}(\mathscr{E}_\mu - \varepsilon_\alpha) \tag{6.38}$$

The quantity ε_α is the excitation energy of the residual nucleus. Since $\mathscr{E}_\mu$ has a negative imaginary part, $P\Psi_\mu$ will grow exponentially with increasing ρ_α. This exponential increase can be interpreted [Humblet and Rosenfeld (61)] as a consequence of the fact that one finds at ρ_α those particles that were emitted by the system at a time $(t - \rho_\alpha/v)$, where v is an average velocity. Since the time

[‡]Siegert (39); Rosenfeld and Humblet (61).

dependence of the state Ψ_μ is

$$e^{-i\mathscr{E}_\mu t/\hbar}$$

this emission occurred when the amplitude of Ψ_μ was larger by the factor $\exp[(\mathrm{Im}\,\mathscr{E}_\mu)\rho_\alpha/v\hbar]$. This behavior is the familiar one; solutions satisfying boundary condition (6.37) are often referred to as the *Gamow solutions*, used by Gamow and Condon and Gurney in their theory of α-decay.

The expansion of the S matrix in terms of its poles was initially carried out by Siegert (39) for the elastic scattering case and was further developed by Humblet and Rosenfeld (61) to take account of reactions generally. However, because of the exponentially diverging nature of the wave functions associated with the poles of the S matrix, the traditional methods of expansion in terms of an orthonormal set are not possible and these authors had recourse to exploiting the analytic properties of the S matrix on the complex energy plane. [For a resolution of the expansion difficulty, see Feshbach (79).]

This problem is avoided, in the formalism presented in this chapter, if $\mathscr{Q}$ contains only *closed* channels. Lemmer and Shakin (64), for example, simply solved the secular equation (2.36), which is just a form of (6.35). The solutions, Λ_μ, are expanded in terms of the closed-channel wave functions so that there is no normalization problem. The Λ_μ's and their adjoint functions $\Lambda_\mu^{(A)}$ form a biorthogonal set:

$$\langle \Lambda_\mu^{(A)} | \Lambda_\nu \rangle = \delta_{\mu\nu} \tag{6.39}$$

We may therefore expand the propagator in (2.30′) to obtain

$$\mathscr{T}_{fi} = \mathscr{T}_{fi}^{(P)} + \sum_\mu \frac{\langle \chi_f^{(-)} | H_{PQ} \Lambda_\mu \rangle \langle \Lambda_\mu^{(A)} | H_{QP} \chi_i^{(+)} \rangle}{E - \mathscr{E}_\mu} \tag{6.40}$$

where $\mathscr{E}_\mu$ and Λ_μ do not depend on the energy E.

This is not an expansion in terms of *all* the S-matrix poles. Referring to (6.3),

$$S = S_P S_R$$

the expansion has been made in terms of poles S_R only, the number being given by the size of the $\mathscr{Q}$ space. The advantages of this procedure are quite manifest. Not only can one obtain an expansion in terms of physically significant matrix elements and energies, but one can also select the energy range over which the expansion is made by selecting the set Φ_s, and one can avoid unimportant poles. It is worthwhile to recall that in single-channel potential scattering, the S_P matrix for a attractive square well has an infinite number of poles. The disadvantage of expansion equation (6.40) is the lack of independence of the various matrix elements and energies $\mathscr{E}_\mu$ because of conditions imposed by unitarity.

D. Microscopic Theory

In a microscopic theory of nuclear reactions, the various quantities, such as the widths and the resonance energies and the matrix elements entering the coupled equations (5.33), (5.34), or (5.35), are all evaluated from "first" principles, namely using a nuclear force and a model of the target and residual nuclei. The analysis of this chapter has been employed in this way by Lemmer and Shakin (64), and Friedman (67) for light nuclear targets. These authors use a shell model or deformed shell model description of the target. Bloch (66) proposed a generalization of the shell model for reaction problems which has been developed by Mahaux and Weidenmüller (69) who have written a treatise entitled *Shell Model Approach to Nuclear Reactions*. We briefly discuss some of these attempts below.

It will be convenient to employ the second quantization formalism of Chapter VII in deShalit and Feshbach (74). We briefly review some of the results we shall need in the present context. In the second quantization formalism, the Hamiltonian is expressed in terms of creation and destruction operators, $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$, respectively, where $\mathbf{r}$, as usual, includes not only spatial but also spin and isospin variables. These operators satisfy the anticommutation relations

$$\{\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}'); \quad \{\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}(\mathbf{r}')\} = \{\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')\} = 0 \tag{6.41}$$

In terms of these operators the number operator, $\hat{N}$, is given by

$$\hat{N} = \int \hat{\psi}^\dagger(\mathbf{r})\psi(\mathbf{r})\, d\mathbf{r} \tag{6.42}$$

An A-particle state is given by

$$|\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_A\rangle = \frac{1}{(A!)^{1/2}} \hat{\psi}^\dagger(\mathbf{r}_A)\hat{\psi}^\dagger(\mathbf{r}_{A-1})\cdots\hat{\psi}^\dagger(\mathbf{r}_2)\hat{\psi}^\dagger(\mathbf{r}_1)|0\rangle \tag{6.43}$$

where the vacuum state $|0\rangle$ satisfies

$$\hat{\psi}(\mathbf{r})|0\rangle = 0$$

The Fock space wave function, with a definite energy E, is given by $\langle E|\mathbf{r}_1\cdots\mathbf{r}_A\rangle$. An $(A+1)$-particle state is given by

$$\frac{1}{\sqrt{A+1}}\hat{\psi}^\dagger(\mathbf{r}_0)|\mathbf{r}_1\cdots\mathbf{r}_A\rangle$$

The normalization of the state given by (6.43) is

$$\langle \mathbf{r}'_1\cdots\mathbf{r}'_A|\mathbf{r}_1\cdots\mathbf{r}_A\rangle = -\frac{1}{A!}\det\delta(\mathbf{r}_\alpha - \mathbf{r}_\beta) \tag{6.44}$$

In problems dealing with the shell model it will be convenient to expand the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ in terms of single-particle wave function φ_k, forming a complete orthonormal set of eigenfunctions defined by

$$(T+U)\varphi_k = \varepsilon_k \varphi_k \tag{6.45}$$

where T is the kinetic energy and U is the shell model potential. Let

$$\hat{\psi}(\mathbf{r}) = \sum a(\mathbf{k})\varphi_k \qquad a(\mathbf{k}) = \langle \varphi_k | \hat{\psi}(\mathbf{r}) \rangle \tag{6.46}$$

Then the coefficients satisfy

$$\begin{gathered} \{a(\mathbf{k}), a^\dagger(\mathbf{k}')\} = \delta(\mathbf{k}, \mathbf{k}'), \qquad \{a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\} = \{a(\mathbf{k}), a(\mathbf{k}')\} = 0 \\ a(\mathbf{k})|0\rangle = 0 \qquad \langle 0|a^\dagger(\mathbf{k}) = 0 \\ \hat{N} = \sum_k a^\dagger(\mathbf{k})a(\mathbf{k}) \end{gathered} \tag{6.47}$$

The operator $K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0)$ can be expressed in terms of the operators $\hat{\psi}$. Note that

$$\begin{aligned} K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0) &\equiv A \int \psi_\nu^*(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_A)\psi_{\nu'}(\mathbf{r}_0, \mathbf{r}_2, \ldots, \mathbf{r}_A) d\mathbf{r}_2 \cdots d\mathbf{r}_A \\ &= A \int \langle \nu | \mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_A \rangle \langle \mathbf{r}_0, \mathbf{r}_2, \ldots, \mathbf{r}_A | \nu' \rangle d\mathbf{r}_2 \cdots d\mathbf{r}_A \end{aligned}$$

Using (6.43) as follows:

$$|\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_A\rangle = \frac{1}{\sqrt{A}} \hat{\psi}^\dagger(\mathbf{r})|\mathbf{r}_2 \cdots \mathbf{r}_A\rangle$$

it follows immediately that

$$K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0) = \langle \nu | \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}_0) | \nu' \rangle \tag{6.48}$$

Inserting expansion (6.46) gives

$$K_{\nu\nu'}(\mathbf{r}, \mathbf{r}_0) = \sum \varphi_{k'}^*(\mathbf{r})\varphi_k(\mathbf{r}_0)\langle k\nu | K | \nu' k' \rangle$$

where

$$\langle k\nu | K | \nu' k' \rangle = \langle \nu | a^\dagger(\mathbf{k}')a(\mathbf{k}) | \nu' \rangle \tag{6.49}$$

The eigenvalue problem for the K operator can now be written as follows:

$$\sum_{\nu', k'} \langle \nu | a^\dagger(\mathbf{k}')a(\mathbf{k}) | \nu' \rangle W_{\nu' k'}^{(\alpha)} = \kappa_\alpha W_{\nu k}^{(\alpha)} \tag{6.50}$$

Equation (6.49) demonstrates again that the set $a_k^\dagger|v\rangle$ is not orthogonal, since the overlap between two such states is given by

$$\langle v|a(\mathbf{k})a^\dagger(\mathbf{k}')|v'\rangle = \delta_{vv'}\delta_{kk'} - \langle \mathbf{k}, v|K|v'\mathbf{k}'\rangle$$

Diagonalizing $1 - K$ is a method for restoring orthogonality, and consequently, removing overcompleteness.

With the use of (6.49) it is a relatively simple matter to determine the matrix, K, for a set of Slater determinants for states v. Friedman (67) provides several examples:

Example 1. The states $|v\rangle$ are single-hole states, N in number, so that

$$|v\rangle = a(\mathbf{k}_i)|\Omega\rangle$$

where Ω is an $(A+1)$-particle state. Then the $N^2 \times N^2$ matrix for K is

$$\begin{aligned}\langle v|a^\dagger(\mathbf{k}_{j'})a(\mathbf{k}_j|v'\rangle &= \langle\Omega|a^\dagger(\mathbf{k}_i)a^\dagger(\mathbf{k}_{j'})a(\mathbf{k}_j)a(\mathbf{k}_{i'})|\Omega\rangle \\ &= \delta(i,i')\delta(j,j') - \delta(i,j)\delta(i',j') \qquad (6.51)\end{aligned}$$

This matrix can readily be diagonalized. The eigenvalues κ_α are found to be unity with a multiplicity $N^2 - 1$, and $1 - N$.

Example 2. The states $|v\rangle$ are single-particle states, N in number, so that

$$|v\rangle = a^\dagger(\mathbf{k}_i)|\Omega\rangle \qquad i = 1,\ldots,N$$

Then

$$\begin{aligned}\langle v|a^\dagger(\mathbf{k}_{j'})a(\mathbf{k}_j)|v'\rangle &= \langle\Omega|a(\mathbf{k}_i)a^\dagger(\mathbf{k}_{j'})a(\mathbf{k}_j)a^\dagger(\mathbf{k}_{i'})|\Omega\rangle \\ &= \delta(j,i')\delta(i,j') \qquad (6.52)\end{aligned}$$

In this case $\kappa_\alpha = 1$ $N(N+1)/2$ times, and $\kappa_\alpha = -1$, $N(N-1)/2$ times.

The problem following (5.23) deals with a simple case of three two-particle states with eight eigenvalues $\kappa_\alpha = 1$ and one equal to -2. Although these results are obtained using Slater determinant wave functions for ψ_v, they can be generalized to a linear combination of the determinants, since such a combination of the determinants is generated by a unitary transformation. As a consequence, the eigenvalues spectrum, κ_α, remains unchanged although the corresponding eigenfunctions of K will be transformed. This is an important remark since it permits the application of the foregoing analysis to more accurate descriptions of the states of the target nuclei, including contributions from unbound orbitals, preserving orthogonality and satisfying the Pauli exclusion principle.

With the eigenfunctions and eigenvalues of K determined, one may now solve the scattering problem using the most convenient of its formulations, (5.33), (5.34), or (5.36). The procedure discussed by Bloch, Mahaux, and Weidenmüller makes two approximations. In the first place it limits the number of nucleons in the continuum of the shell model potential to one, thus restricting the application to scattering (including charge exchange) reactions and not permitting the treatment of particle exchange reactions. This is an approximation even for scattering processes since states with two or more particles in the continuum may be of importance, indeed are, for accurate descriptions of the target wave functions, as mentioned above. The second approximation, not completely independent of the first, limits the description of the target (A particle) states and those of the $(A+1)$ compound system to linear combinations of Slater determinants formed from bound single-particle wave functions only.

A set of states for the full $(A+1)$-particle system is constructed. The "bound" states Φ_i are shell model states, which like the target states consist of linear combinations of Slater determinants constructed from $A+1$ bound single-particle levels. These are finite in number. The energy of these states can be above the threshold for particle emission so that they will generate resonances as described in the introduction to this chapter. The scattering states of the $(A+1)$ system, $\chi_c(E)$, are constructed by antisymmetrizing the product of the wave function for a particle in the continuum and a target wave function. Because of the special construction of all the states from single-particle wave functions which are eigenfunctions of a common energy-independent single-particle Hamiltonian, these scattering states, together with the bound states, form an orthogonal set which we will assume is appropriately normalized. We shall refer to them as the *shell model states.* Thus by making the assumptions listed above, the Pauli principle and problems of overcompleteness are avoided.

The formalism developed by Mahaux and Weidenmüller (69) can be expressed in terms of projection operators. These are explicitly constructed from the $(A+1)$-particle shell model states. The operator Q projects on to the set Φ_n, while P projects on to the shell model scattering states χ_c.

$$P = \sum_{c'} \int dE' \chi_c(E') \rangle \langle \chi_c(E') \tag{6.53}$$

$$Q = \sum_{n} \Phi_n \rangle \langle \Phi_n \tag{6.54}$$

These are of course not identical with the P and Q used in Section 5. However, the general analysis of Section 2 is applicable. Equations (2.3) and (2.4) become

$$[E - (H_0)_{PP} - V_{PP}] P\Psi = V_{PQ} Q\Psi \tag{6.55}$$

$$[E - (H_0)_{QQ} - V_{QQ}] Q\Psi = V_{QP} P\Psi \tag{6.56}$$

where the full Hamiltonian H is

$$H = H_0 + V$$

and H_0 is the shell model Hamiltonian.

For further discussion of this model the reader is referred to Mahaux and Weidenmüller's book (69). It is clear that its limitation to one particle in the continuum precludes a realistic description of a compound nuclear resonance and *a fortiori* of transfer reactions. It can be used to describe doorway states, particularly those which are constructed from 1p–1h states excited, for example, by photon absorption [Bloch and Gillet (65)]. This becomes a valid description if V is complex, to allow for energy averaging over compound nuclear resonances. The model is very useful as well in providing a description of the structure of the S matrix, which as we discussed earlier, is insensitive to dynamical details.

7. SUMMARY

In this chapter a formal theory of nuclear reactions is developed, based on a separation of the channels of the system into prompt and time-delaying states. This is accomplished formally through the use of projection operators P and Q. Without specifying any but general properties of these operators, a theory in which direct reactions and compound nuclear resonances appear simultaneously and on equal footing is developed in Section 2. Both the case of an isolated resonance and that of overlapping resonances are treated. Still maintaining the generality of P and Q, the energy average of the transition amplitude is discussed in Section 3 and a derivation of the optical model exhibited. It should be remembered that this optical model is not necessarily the single-channel optical model but can include many channels, so that it provides not only a description of elastic scattering but also one of inelastic scattering and direct processes generally. The theory of the doorway state and its relationship to intermediate structure in the cross section is the subject of Section 4. Exit as well as entrance doorways are treated, as well as the effect of a doorway resonance on the fine structure resonances. In Section 5 we consider a more specific P together with the problems of overcompleteness, the Pauli principle, and the lack of orthogonality which occur if the prompt wave function is expanded as a finite series in the wave functions of the target nucleus, as is appropriate for the discussion of elastic and inelastic scattering. These problems, as encountered more seriously in transfer reactions, will be discussed in a later chapter. Finally, in the last section the general methods (Section 2 particularly) developed in this chapter are compared with other formalisms, including that of Kapur–Peierls, Wigner and Eisenbud, and Bloch, Mahaux, and Weidenmüller, Many contributions to our understanding of nuclear reactions that are appropriate to this chapter have not been discussed. There

is the work of Macdonald, Tobocman and his collaborators, Danos and Greiner, Thaler, Shakin, and many others. Of course, no attempt has been made to provide a historical perspective.

Reactions induced (or produced) by γ-rays can readily be included without modification of the general development. As expected, the width of a resonance, Γ, will now contain a component, Γ_γ, because of the possible radiative decay of a resonance. In addition, direct γ-ray as well as doorway state processes are automatically described by the theory, which is discussed by Estrada [Estrada and Feshbach (63)].

APPENDIX. THE BOUNDARY CONDITION MODEL FOR NUCLEAR REACTIONS‡

In the boundary condition model, configuration space is divided into two regions, one in which the strong nuclear interactions prevail, $r < R$, and one in which the potential is zero (for neutrons) or Coulombic (for charged particles). R is, roughly speaking, the nuclear radius. No such sharp boundary exists in nature. But the model is useful because it isolates the major physical effects, the corresponding parameters, and qualitative and quantitative estimates of their numerical values. The model was developed in a simple form by Feshbach, Peaslee, and Weisskopf (47) and in a more complete and sophisticated form by Wigner and Eisenbud (47). It has been used by Lomon and Feshbach (68) to study the nucleon–nucleon interaction, by Lomon (89) and by Jaffe and Low (79) for elementary particle reactions.

In this section we use the simpler description of Feshbach et al. (47). Spin is neglected. A partial wave expansion is assumed. The radial wave function for the lth partial wave, corresponding to an orbital angular momentum of $l\hbar$, is given by $(1/r)\psi_l$. The boundary condition to be met by the external wave function $(r > R)$ at $r = R$ is

$$f_l = R\left[\frac{d\psi_l/dr}{\psi_l}\right]_{r=R} \tag{A.1}$$

where f_l can be a complex function of the energy. The resultant S matrix is

$$S_l \equiv e^{2i\delta_l} = \frac{w_l^{(-)} f_l/kR - w_l^{(-)\prime}/w_l}{w_l^{(+)} f_l/kR - w_l^{(+)\prime}/w_l} \tag{A.2}$$

where $w_l^{(\pm)}$ are solutions of the Schrödinger equation for $r > R$ evaluated at $r = R$. The prime denotes derivative. The functions $w_l^{(\pm)}$ for uncharged particles

‡Feshbach, Peaslee, and Weisskopf (47).

have the following asymptotic dependence:

$$w_l^{(+)}(\zeta) \underset{\zeta \to \infty}{\longrightarrow} e^{i(\zeta - l\pi/2)} \tag{A.3}$$

$$w_l^{(-)}(\zeta) \underset{\zeta \to \infty}{\longrightarrow} e^{-i(\zeta - l\pi/2)}$$

$$\zeta = kr$$

Let

$$w_l^{(+)} = |w_l| e^{-i\sigma_l} \, w_l^{(+)\prime} = |w_l'| e^{-i\tau_l} \tag{A.4}$$

Then

$$|w_l| \, |w_l'| \sin(\sigma_l - \tau_l) = 1 \tag{A.5}$$

Let

$$\frac{w^{(+)\prime}}{w_l^{(+)}} \equiv \Delta_l + i s_l \tag{A.6}$$

Then

$$\Delta_l = \frac{|w_l'|}{|w_l|} \cos(\sigma_l - \tau_l) \tag{A.7}$$

$$s_l = \frac{1}{|w_l|^2} \tag{A.8}$$

The scattering matrix can then be written

$$S_l = e^{2i\sigma_l} \frac{(f_l/kR) - \Delta_l + i s_l}{(f_l/kR) - \Delta_l - i s_l} \tag{A.9}$$

The transmission factor T_l is given by

$$T_l = 1 - |S_l|^2 = -4 \frac{\operatorname{Im} f_l (kR s_l)}{(\operatorname{Re} f_l - kR\Delta_l)^2 + (kR s_l - \operatorname{Im} f_l)^2} \tag{A.10}$$

Note that

$$T_l \geqslant 0 \qquad \text{and} \qquad \operatorname{Im} f_l \leqslant 0 \tag{A.11}$$

Resonance Formula

Consider first $l = 0$, for which $\Delta_0 = 0$. Let E_0 be the value for which $\operatorname{Re} f_0 = 0$. Then in the neighborhood of $E = E_0$,

$$T_0 = \frac{\Gamma_i \Gamma_0}{(E - E_0)^2 + \Gamma^2/4} \tag{A.12}$$

and

$$S_0 - 1 = (e^{2i\sigma_0} - 1) - e^{2i\sigma_0} \frac{i\Gamma_0}{(E - E_0) + i\Gamma/2} \tag{A.13}$$

where

$$\Gamma = \Gamma_0 + \Gamma_i \tag{A.14}$$

$$\Gamma_i = \frac{2|\operatorname{Im} f_0|}{|\partial \operatorname{Re} f_0/\partial E|_{E=E_0}} \tag{A.15}$$

$$\Gamma_0 = \frac{2kR}{|\partial \operatorname{Re} f_0/\partial E|_{E=E_0}} \tag{A.16}$$

If the interaction region is a square well,

$$V = -V_0 - iW_0 \qquad r \leqslant R$$

then at resonance

$$\left(\frac{\partial \operatorname{Re} f_0}{\partial E}\right)_{E=E_0} = -\frac{\mu R^2}{\hbar^2}$$

$$\operatorname{Im} f_0 = -\frac{\mu R^2}{\hbar^2} W_0$$

and

$$T_0 = \frac{4W_0(\hbar^2/\mu R^2)kR}{(E - E_0)^2 + [W_0 + (\hbar^2/\mu R^2)kR]^2} \tag{A.17}$$

These results are easily generalized for $l \neq 0$. We find that

$$T_l = \frac{\Gamma_i^{(l)}\Gamma_0^{(l)}}{(E - E_0)^2 + \frac{1}{4}(\Gamma_l)^2} \tag{A.18}$$

where E_0 is the energy at which $\operatorname{Re} f_l - kR\Delta_l = 0$, and

$$\Gamma_l = \Gamma_i^{(l)} + \Gamma_0^{(l)} \tag{A.19}$$

$$\Gamma_i^{(l)} = \frac{2|\operatorname{Im} f_l|}{|\partial(\operatorname{Re} f_l - kRs_l)/\partial E|_{E=E_0}} \tag{A.20}$$

$$\Gamma_0^{(l)} = \frac{2kRs_l}{|\partial(\operatorname{Re} f_l - kRs_l)/\partial E|_{E=E_0}} \tag{A.21}$$

The penetrability P_l is defined to be

$$P_l = kRs_l = \frac{kR}{|w_l|^2} \tag{A.22}$$

$\mathscr{R}$ Matrix ($l=0$)

Let X_λ be the solutions for the interior ($r < R$) problem satisfying

$$X'_\lambda(R) \equiv \left(\frac{\partial X_\lambda}{\partial r}\right)_{r=R} = 0$$

The eigenvalues will form a discrete spectrum. The energy accompanying the eigenfunction X_λ is E_λ. We now expand the exact solution ψ in terms of X_λ in the region $r < R$. We assume that on the surface $r = R$:

$$\psi = \sum_c \chi_c(s)\phi_c(r)$$

where $\chi_c(s)$ forms an orthonormal set on the surface s, $r = R$. It then follows that

$$\phi_c(R) = -\frac{\hbar^2}{2\mu}\sum_{c'} \phi'_{c'}(R)\sum_\lambda \frac{1}{E - E_\lambda}\left[\int ds\, X_\lambda \chi_{c'}(s)\right]\left[\int ds\, X_\lambda \chi_c^*(s)\right]$$

The $\mathscr{R}$ matrix is therefore

$$\mathscr{R}_{cc'} = \sum_\lambda \frac{1}{E_\lambda - E}\gamma^*_{\gamma c}\gamma_{\lambda c'} \tag{A.23}$$

where

$$\gamma_{\lambda c} = \sqrt{\frac{\hbar^2}{2\mu}}\int X_\lambda \chi_c dS \tag{A.24}$$

so that

$$\phi_c(R) = \sum_{c'} R_{cc'}\phi'_{c'}(R) \tag{A.25}$$

For the single-channel case, $R_{cc'} = R_{cc}\delta_{cc'}$,

$$\phi_c(R) = R_{cc}\phi'_{c'}(R)$$

so that the f of (A.1) is

$$f_0 = \frac{R}{\mathscr{R}} \tag{A.26}$$

Properties of s_l and Δ_l

$$s_l(\zeta) \xrightarrow[\zeta \to 0]{} \left(\frac{\zeta^l}{(2l-1)!!}\right)^2 \qquad \Delta_l \xrightarrow[\zeta \to 0]{} -\frac{l}{\zeta}$$

$$\xrightarrow[\zeta \to \infty]{} 1 \qquad \xrightarrow[\zeta \to \infty]{} O\left(\frac{1}{\zeta^3}\right) \tag{A.27}$$

$$s_0 = 1 \qquad \Delta_0 = 0$$

$$s_1 = \frac{\zeta^2}{(1+\zeta^2)} \qquad \zeta\Delta_1 = -\frac{1}{(1+\zeta^2)} \tag{A.28}$$

$$s_2 = \frac{\zeta^4}{(9+3\zeta^2+\zeta^4)} \qquad \zeta\Delta_2 = -\frac{3(6+\zeta^2)}{9+3\zeta^2+\zeta^4}$$

For charged particles,

$$w_l^{(+)} = G_l + iF_l \tag{A.29}$$

The properties of F_l and G_l are given by (A.70)–(A.74) in Appendix A at the end of the book.