APPENDIX A

SCATTERING THEORY

H AND V HERMITIAN

Lippman–Schwinger Equation

$$\psi^{(+)} = \phi + \frac{1}{E^{(+)} - H_0} V\psi^{(+)} \tag{1}$$

$$= \phi + \frac{1}{E^{(+)} - H} V\phi \tag{2}$$

where

$$H_0\phi = E\phi$$

and

$$E^{(+)} = E + i\eta \qquad \eta \to 0^+$$

$$\psi^{(-)} = \phi + \frac{1}{E^{(-)} - H_0} V\psi^{(-)} \tag{3}$$

$$= \phi + \frac{1}{E^{(-)} - H} V\phi \tag{4}$$

where

$$E^{(-)} = E - i\eta \qquad \eta \to 0^+$$

$$\psi^{(+)}(\mathbf{k}_i, \mathbf{r}) = \psi^{(-)*}(-\mathbf{k}_i, \mathbf{r}) \tag{5}$$

$\mathscr{T}$ Matrix

$$\mathscr{T}_{fi} = \langle \psi_f^{(-)} V \phi_i \rangle = \langle \phi_f \mathscr{T}(E^{(+)}) \phi_i \rangle \tag{6}$$

$$= \langle \phi_f V \psi_i^{(+)} \rangle \qquad \text{when } E_f = E_i \tag{7}$$

$$\mathscr{T}(E) = V + V \frac{1}{E - H_0} \mathscr{T}(E) \tag{8}$$

$$= V + V \frac{1}{E - H} V \tag{9}$$

Reaction Probability/Time

$$w_{fi} = \frac{2\pi}{\hbar} |\mathscr{T}_{fi}|^2 \delta(E_f - E_i)$$

For comparison with experiment, one must average over the initial states (e.g., spin) and sum over final states. The sum over final energies is the integral

$$d\bar{w}_{fi} = \int \rho_f \, dw_{fi} \, dE_f = \frac{2\pi}{\hbar} |\mathscr{T}_{fi}|^2 \rho_f \qquad E_f = E_i \tag{10}$$

where ρ_f is the density of states at E_f and $\phi_{f,i} = \exp[i(\mathbf{k}_{f,i} \cdot \mathbf{r})]$.

Cross Section

$$\frac{d\sigma}{d\Omega} = \frac{w_{fi}}{j_i}$$

where j_i is the incident current.

Let $H = H_0 + V_0 + V_1$; then

$$\mathscr{T}_{fi} = \langle \phi_f V_0 \chi_i^{(+)} \rangle + \langle \chi_f^{(-)} V_1 \psi_i^{(+)} \rangle \tag{11}$$

where

$$(H_0 + V_0)\chi = E\chi$$

Single Channel

$$\rho_f = \left(\frac{1}{2\pi}\right)^3 k_f^2 \frac{dk_f}{dE} = \left(\frac{1}{2\pi}\right)^3 \frac{\mu}{\hbar^2} k_f \tag{12}$$

where

$$\mu = \text{reduced mass}$$

$$j_i = \frac{\hbar k_i}{\mu} \tag{13}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\frac{\mu^2 k_f}{\hbar^4 k_i}|\mathscr{T}_{fi}|^2$$

Scattering Amplitude

$$f_{fi} = -\frac{1}{2\pi}\frac{\mu}{\hbar^2}\mathscr{T}_{fi} \tag{14}$$

The appearance of these formulas depends on the normalization of ϕ_i. For example, let

$$\phi_i = \sqrt{\rho_i}e^{i\mathbf{k}_i\cdot\mathbf{r}} \qquad \text{and} \qquad \phi_f = \sqrt{\rho_f}e^{i\mathbf{k}_f\cdot\mathbf{r}}$$

Then

$$\bar{w}_{fi} = \frac{2\pi}{\hbar}|\mathscr{T}_{fi}|^2$$

and

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4}{k_i^2}|\mathscr{T}_{fi}|^2 \tag{15}$$

It is important to ascertain the normalization of $\phi_{i,f}, \psi_i^{(+)}$, and $\psi_f^{(-)}$.

S Matrix

$$S_{fi} = \langle \psi_f^{(-)}, \psi_i^{(+)} \rangle = \langle \phi_f, S\phi_i \rangle \tag{16}$$

$$SS^\dagger = S^\dagger S = 1 \tag{17}$$

$$S = 1 - 2\pi\mathscr{T}\delta(E_f - E_i) \tag{18}$$

$$\mathscr{T}_{if}^* - \mathscr{T}_{fi} = 2\pi i \sum_g \mathscr{T}_{gf}^*\mathscr{T}_{gi} \qquad (E_f = E_i = E)$$

Single-Channel

$$\mathscr{T}^*(\mathbf{k}_i, \mathbf{k}_f) - \mathscr{T}(\mathbf{k}_f, \mathbf{k}_i) = 2\pi i\rho \int \mathscr{T}^*(\mathbf{k}_g, \mathbf{k}_f)\mathscr{T}(\mathbf{k}_g, \mathbf{k}_i)\,d\Omega_g \tag{19}$$

or

$$f(\mathbf{k}_f, \mathbf{k}_i) - f^*(\mathbf{k}_i, \mathbf{k}_f) = \frac{ik}{2\pi} \int f^*(\mathbf{k}_g, \mathbf{k}_f) f(\mathbf{k}_g, \mathbf{k}_i) d\Omega g \tag{20}$$

where in both Eq. (19) and Eq. (20)

$$E_f = E_i = E$$

Generally,

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \operatorname{Im} f(0^\circ) \tag{21}$$

This equation holds even when H is not Hermitian.

$\mathscr{K}$ Matrix

Let

$$\psi^{(0)} = \phi + \mathscr{P} \frac{1}{E - H_0} V \psi^{(0)}$$

where $\mathscr{P}$ symbolizes the principal value of the integral.

$$\mathscr{P} \frac{1}{E - H_0} = \frac{1}{2} \left[\frac{1}{E^{(+)} - H_0} + \frac{1}{E^{(-)} - H_0} \right]$$

$$\mathscr{K}_{ba} \equiv \langle \phi_b, V \psi_a^{(0)} \rangle \tag{22}$$

The operator $\mathscr{K}$ is defined by

$$\mathscr{K}_{ba} = \langle \phi_b, \mathscr{K} \phi_a \rangle \tag{23}$$

$$\mathscr{K} = V + V \frac{\mathscr{P}}{E - H_0} \mathscr{K} \tag{24}$$

$$\psi^{(0)} = \psi^{(+)}(1 + i\pi\mathscr{K}) \tag{25}$$

$$\psi^{(+)} = \psi^{(0)}(1 - i\pi\mathscr{T}) \tag{26}$$

$$\mathscr{K} = \mathscr{T} + i\pi\mathscr{T}\mathscr{K} = \mathscr{T} + i\pi\mathscr{K}\mathscr{T} \qquad \text{Heitler integral equation} \tag{27}$$

or

$$\mathscr{T} = \frac{\mathscr{K}}{1 + i\pi\mathscr{K}} \tag{28}$$

$$S = \frac{1 - i\pi\mathscr{K}}{1 + i\pi\mathscr{K}} \tag{29}$$

K is Hermitian,

$$\mathscr{K}_{ba} = \langle \phi_b, V_0 \chi_a^{(0)} \rangle + \langle \chi_b^{(0)}, V_1 \psi_a^{(0)} \rangle \tag{30}$$

where

$$\chi_a^{(0)} = \phi_a + \frac{\mathscr{P}}{E - H_0} V_0 \chi_a^{(0)} = \phi_a + \frac{\mathscr{P}}{E - H_0 - V_0} V_0 \phi_a$$

$$\psi_a^{(0)} = \phi_0 + \frac{\mathscr{P}}{E - H_0 - V_0} V_1 \psi_a^{(0)}$$

Dispersion Relations

Subject to the condition that $\int |V(\mathbf{r})| d\mathbf{r}$ is bounded

$$\operatorname{Re} f(0^\circ) = f_{\text{Born}}(0^\circ) - \frac{2\pi\hbar^2}{\mu} \sum \frac{N_b^2}{E + \varepsilon_b} + \frac{1}{4\pi^2} \mathscr{P} \int_0^\infty \frac{k' \sigma_{\text{tot}}(E')}{E' - E} dE' \tag{31}$$

where ε_b are the binding energy of the bound states whose normalized wave function asymptotically satisfy

$$\psi_b \to N_b e^{-\kappa_b r / r} \qquad \kappa_b = \sqrt{\frac{2\mu}{\hbar^2} \varepsilon_b} \tag{32}$$

H AND V NOT HERMITIAN[‡]

Equation (4) is replaced by

$$\psi^{(-)} = \phi + \frac{1}{E^{(-)} - H^\dagger} V^\dagger \phi \tag{33}$$

In addition, two new functions need to be defined:

$$\tilde{\psi}^{(+)} = \phi + \frac{1}{E^{(+)} - H^\dagger} V^\dagger \phi \tag{34}$$

$$\tilde{\psi}^{(-)} = \phi + \frac{1}{E^{(-)} - H} V \phi \tag{35}$$

The function $\tilde{\psi}^{(+)}$ is the wave function for a regenerative potential $V^\dagger$ of V is absorptive. We note that

$$\langle \tilde{\psi}_a^{(+)}, \psi_b^{(+)} \rangle = \delta_{ab} \qquad \langle \psi_a^{(-)}, \tilde{\psi}_b^{(-)} \rangle = \delta_{ab} \tag{36}$$

[‡]Feshbach (85).

The S matrix remains as defined by (16). We define

$$\tilde{S}_{ba} = \langle \tilde{\psi}_b^{(+)}, \tilde{\psi}_a^{(-)} \rangle = \langle \phi_b, \tilde{S}\phi_a \rangle \tag{37}$$

Then

$$S\tilde{S} = \tilde{S}S = 1 \tag{38}$$

If

$$\tilde{\mathscr{T}}_{ba} = \langle \tilde{\psi}_b^{(+)} V\phi_a \rangle = \langle \phi_b V\tilde{\psi}_a^{(-)} \rangle \qquad E_a = E_b$$

then

$$\tilde{S}_{ba} = \delta_{ba} + 2\pi i\delta(E_a - E_b)\tilde{\mathscr{T}}_{ba} \tag{39}$$

$$\tilde{\mathscr{T}} - \mathscr{T} = 2\pi i\mathscr{T}\tilde{\mathscr{T}} = 2\pi i\tilde{\mathscr{T}}\mathscr{T} \tag{40}$$

If

$$\mathscr{T}^{(e)} = \tfrac{1}{2}(\tilde{\mathscr{T}} - \mathscr{T})$$

and

$$\mathscr{T}^{(0)} = \tfrac{1}{2}(\tilde{\mathscr{T}} + \mathscr{T})$$

then for $q = 0$,

$$\mathscr{T}^{(0)} = \mathscr{T}_{\mathrm{Born}} + \frac{1}{\pi i}\mathscr{P}\int \frac{dE'}{E - E'}\mathscr{T}^{(e)}(E') + \text{sum over bound states}$$

Let

$$\tilde{\mathscr{K}}_{ba} = \langle \phi_b V\tilde{\psi}_a^{(+)} \rangle = \langle \phi_b \tilde{\mathscr{K}}\phi_a \rangle \tag{41}$$

Then

$$\tilde{\mathscr{K}} = \mathscr{K}$$

$$\tilde{S} = \frac{1 + i\pi\mathscr{K}}{1 - i\pi\mathscr{K}} \tag{42}$$

$\mathscr{K}$ is not Hermitian.

Single Channel

$$\psi^{(+)} \xrightarrow[r\to\infty]{} \frac{1}{r}(e^{-ikr} - S(k)e^{ikr})$$

$$\tilde{\psi}^{(-)} \xrightarrow[r\to\infty]{} \frac{1}{r}(e^{ikr} - \tilde{S}(k)e^{-ikr})$$

For a potential $V + iW$,

$$f(\mathbf{k}_f, \mathbf{k}_i) - f^*(\mathbf{k}_i, \mathbf{k}_f) = \frac{ik}{2\pi}\int d\Omega_g f^*(\mathbf{k}_g, \mathbf{k}_f) f(\mathbf{k}_g, \mathbf{k}_i)$$

$$- \frac{i}{2\pi}\int \psi^{(+)*}(\mathbf{k}_f, \mathbf{r}) \frac{2\mu}{\hbar^2} W \psi^{(+)}(\mathbf{k}_i, \mathbf{r})\, d\mathbf{r} \qquad (44)$$

Phase-Shift Analysis

Scattering of a spinless particle by a spinless nucleus. Central potential

$$f(\vartheta, \varphi) = \frac{1}{2ik}\sum (2l+1) P_l(\cos\vartheta)(e^{2i\delta_l} - 1) \qquad (45)$$

$$\langle l'm' | S | lm \rangle = S_l \delta(l, l') S(m, m')$$

$$S_l = e^{2i\delta_l} \qquad (46)$$

$$\mathcal{T}_l = -\frac{1}{\pi} e^{i\delta_l} \sin\delta_l \qquad (47)$$

$$\sigma_{\mathrm{el}} = \frac{4\pi}{k^2}\sum_l (2l+1)|S_l - 1|^2 \qquad (48)$$

$$\sigma_a = \frac{\pi}{k^2}\sum (2l+1)[1 - |S_l|^2] \qquad (49)$$

If $\delta_l = \xi_l + i\eta_l$,

$$|S_l - 1|^2 = 1 - 2e^{-2\eta_l}\cos 2\xi_l + e^{-4\eta_l} \qquad (50)$$

$$1 - |S_l|^2 = 1 - e^{-4\eta_l} \qquad (51)$$

Transmission Factor

$$T_l = 1 - |S_l|^2 \qquad (52)$$

Zero-Energy Limit (Uncharged Particles)

$$S_l \to O(k^{2l+1}) \qquad (53)$$

$$T_l \to O(k^{2l+1}) \qquad (54)$$

Effective Range

$$k^{2l+1}\cot\delta_l = -\frac{1}{a_l} + \frac{1}{2}k^2 r_l \qquad (55)$$

$a_l \equiv$ scattering length

$r_l =$ effective range

For $l=0$, a_0 and r_0 have the dimension of a length. $l=1$, a_l, and r_l have the dimensions of a volume. In Chapter X on kaon–nucleus interactions, the minus sign on the right-hand side (55) is dropped.

Effective Volume

$$\frac{\tan\delta_l}{k^{2l+1}} = -a_l - \tfrac{1}{2}k^2 v_l \tag{56}$$

For $l=0$, v_l has the dimensions of a volume.

Elastic Scattering of a Spin Zero by a Spin-$\frac{1}{2}$ System

$$\hat{f}(k,\vartheta) = A(k,\vartheta) + B(k,\vartheta)\boldsymbol{\sigma}\cdot\mathbf{n}$$

$$\mathbf{n} = \frac{\mathbf{k}_i \times \mathbf{k}_f}{|\mathbf{k}_i \times \mathbf{k}_f|} \tag{57}$$

where $\hbar\mathbf{k}_i$ = incident momentum and $\hbar\mathbf{k}_f$ = final momentum.

$$A(k,\vartheta) = \frac{1}{2ik}\sum_0^\infty [(l+1)(e^{2i\delta_l^{(+)}}-1) + l(e^{2i\delta_l^{(-)}}-1)]P_l(\cos\vartheta)$$

$$B(k,\vartheta) = \frac{1}{2k}\sum_1^\infty (e^{2i\delta_l^{(+)}} - e^{2i\delta_l^{(-)}})P_l^{(1)}(\cos\vartheta) \tag{58}$$

$$P_l^{(1)} = \sin\vartheta \frac{d}{d(\cos\vartheta)}P_l(\cos\vartheta)$$

where asymptotically ($r\to\infty$),

$$\psi(j=l\pm\tfrac{1}{2}) \to \sin\left(kr - \frac{l\pi}{2} + \delta_l^{(\pm)}\right) \tag{59}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = |A|^2 + |B|^2 \tag{60}$$

$$\sigma_{\text{tot}}^{\text{el}} = \frac{\pi}{k^2}\sum_{l=0}^\infty \{(l+1)|e^{2i\delta_l^{(+)}}-1|^2 + l|e^{2i\delta_l^{(-)}}-1|^2\} \tag{61}$$

$$\mathbf{P} = \text{polarization} = \frac{\operatorname{tr}\hat{f}^\dagger\boldsymbol{\sigma}f}{\operatorname{tr}\hat{f}^\dagger\cdot\hat{f}} = 2\frac{\operatorname{Re}AB^*}{|A|^2+|B|^2} = 2P\mathbf{n} \tag{62}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}}[1 + P_p P\mathbf{n}_p\cdot\mathbf{n}] \tag{63}$$

where $P_p\mathbf{n}_p$ is the beam polarization,

$$Q = \frac{2\,\mathrm{Im}\,AB^*}{|A^2| + |B|^2} \tag{64}$$

Coulomb Wave Function: Elastic Scattering

$$\left(\nabla^2 + k^2 - \frac{2\eta k}{r}\right)\psi(\mathbf{r}) = 0 \qquad \eta = \frac{Z_1 Z_2 e^2}{\hbar v} \qquad v = \text{velocity}$$

$$\psi(\mathbf{r}) = \Gamma(1 + i\eta)e^{-(1/2)\pi\eta}e^{ikz}F(-i\eta; 1; ik(r - z))$$

$$F(a, b; \xi) = 1 + \frac{a}{b}\xi + \frac{1}{2!}\frac{a(a+1)}{b(b+1)}\xi^2 + \cdots \tag{65}$$

$$\psi \to e^{i[kz + \eta \ln k(r-z)]} - \frac{\eta}{k(r-z)}\frac{\Gamma(1 + i\eta)}{\Gamma(1 - i\eta)}e^{i(kr - \eta \ln k(r-z))} \tag{66}$$

$$f = -\frac{\eta}{2k\sin^2\frac{1}{2}\vartheta}e^{-i\eta \ln \sin^2(1/2)\vartheta + 2i\sigma_0} \tag{67}$$

$$e^{2i\sigma_0} = \frac{\Gamma(1 + i\eta)}{\Gamma(1 - i\eta)} \tag{68}$$

$$\text{Rutherford cross section} = \frac{\eta^2}{4k^2}\sin^4\frac{\vartheta}{2} \tag{69}$$

Partial Wave Expansion

$$\psi_{lm} = Y_{lm}\frac{F_l(r)}{kr}$$

where F_l = regular spherical Coulomb wave function,

$$F_l'' + \left[k^2 - \frac{2\eta k}{r} - \frac{l(l+1)}{r^2}\right]F_l = 0$$

$$F_l = \frac{2^l e^{-(1/2)\pi\eta}|\Gamma(l + 1 + i\eta)|}{(2l+1)!}e^{-ikr}(kr)^{l+1}F(l + 1 - i\eta; 2l + 2; 2ikr)$$

$$\to \sin\left(kr - \eta \ln 2kr - \frac{l\pi}{2} + \sigma_l\right) \qquad kr \to \infty$$

$$\to C_l(kr)^{l+1}$$

$$e^{2i\sigma_l} = \frac{\Gamma(l + 1 + i\eta)}{\Gamma(l + 1 - i\eta)} \tag{70}$$

$$\psi = \frac{1}{kr}\Sigma(2l+1)i^l e^{i\sigma_l}F_l(r)P_l(\cos\theta) \tag{71}$$

G_l = irregular spherical Coulomb wave function

$$\rightarrow \cos\left(kr - \eta\ln 2kr - \frac{l\pi}{2} + \sigma_l\right) \qquad kr \rightarrow \infty$$

$$G_l \rightarrow \frac{1}{(2l+1)C_l}(kr)^{-l}\left[1 + \begin{cases} O(\eta kr \ln kr) & l = 0 \\ O\left(\frac{\eta}{l}kr\right) & l \neq 0 \end{cases}\right] \qquad kr \rightarrow 0$$

$$C_l = \frac{2^l e^{-\pi\eta/2}|\Gamma(l+1+i\eta)|}{(2l+1)!} \qquad \sigma_l = \sigma_0 + \sum_{t=1}^{l}\tan^{-1}\frac{\eta}{t} \tag{72}$$

$$C_0^2 = \frac{2\pi\eta}{e^{2\pi\eta}-1} \tag{73}$$

$$C_l^2 = \frac{(1+\eta^2)(4+\eta^2)\cdots(l^2+\eta^2)2^{2l}}{(2l+1)^2[(2l)!]^2}C_0^2 \tag{74}$$

For properties of these functions and numerical values, see Chapter 14 (p. 537 et seq.) of *Handbook of Mathematical Functions*, M. Abramovitz and I. A. Stegen, eds., National Bureau of Standards (U.S. Government Printing Office, Washington, D.C., 1964).

The solutions of the field-free equations

$$\frac{d^2u}{d\zeta^2} + \left[1 - \frac{l(l+1)}{\zeta^2}\right]u = 0 \tag{75}$$

of interest are

$$u_l = \zeta j_l(\zeta)$$

$$v_l = \zeta n_l(\zeta)$$

$$w_l^{(+)} = v_l + iu_l = i\zeta h_l(\zeta) \tag{76}$$

where j_l, n_l, and h_l are the spherical Bessel, Neumann, and Hankel functions:

$$u_l \xrightarrow[\zeta\to 0]{} \frac{\zeta^{l+1}}{(2l+1)!!} \qquad v_l \xrightarrow[\zeta\to 0]{} \frac{(2l-1)!!}{\zeta^l} \tag{77}$$

$$\xrightarrow[\zeta\to\infty]{} \sin\left(\zeta - \frac{l\pi}{2}\right) \qquad \xrightarrow[\zeta\to\infty]{} \cos\left(\zeta - \frac{l\pi}{2}\right) \tag{78}$$

$$w_l \xrightarrow[\zeta\to 0]{} v_l \tag{79}$$

$$\xrightarrow[\zeta\to\infty]{} e^{i(\zeta - l\pi/2)} \tag{80}$$

$$u_0(\zeta) = \sin\zeta \qquad v_0 = \cos\zeta$$

$$u_1(\zeta) = \frac{\sin\zeta}{\zeta} - \cos\zeta \qquad v_1 = \frac{\cos\zeta}{\zeta} + \sin\zeta$$

$$u_2(\zeta) = \left(\frac{3}{\zeta^2} - 1\right)\sin\zeta - \frac{3}{\zeta}\cos\zeta \qquad v_2 = \left(\frac{3}{\zeta^2} - 1\right)\cos\zeta + \frac{3}{\zeta}\sin\zeta \tag{81}$$

$$w_0^{(+)} = e^{i\zeta} \qquad w_1^{(+)} = \frac{1 - i\zeta}{\zeta}e^{i\zeta} \qquad w_2^{(+)} = \frac{3 - 3i\zeta - \zeta^2}{\zeta^2}e^{i\zeta} \tag{82}$$

Recurrence relations satisfied by u_l, v_l, and $w_l^{(+)}$:

$$\frac{2l+1}{\zeta}u_l = u_{l-1} + u_{l+1} \qquad (2l+1)u_l' = (l+1)u_{l-1} - lu_{l+1} \tag{83}$$

APPENDIX B

CROSS SECTIONS; VECTOR AND TENSOR POLARIZATIONS‡

In the following it will be assumed that the target has zero spin and the projectile *a* spin of S. This is a useful choice since many of the target nuclei have zero spin. But it is also totally general since it applies to the channel spin representation, in which the spin of the projectile and target are combined and then combined with the relative orbital angular momentum, $\mathbf{l}$:

$$\mathbf{S} = \mathbf{S}_T + \mathbf{S}_P$$

and

$$\mathbf{J} = \mathbf{l} + \mathbf{S}$$

$\mathbf{S}_T$ and $\mathbf{S}_P$ are the target and projectile spins, respectively.

Let the reaction amplitude be

$$f(\alpha_i S M_i \rightarrow \alpha_f S M_f) = \langle S M_f | \hat{F}(\alpha_f, \alpha_i) | S M_i \rangle \tag{1}$$

In this equation M_i and M_f are components of $\mathbf{S}$ along an arbitrary direction. The parameters α and α' are the remaining quantum numbers needed to specify the initial and final states, including the relative momenta $\hbar\mathbf{k}_f$ and $\hbar\mathbf{k}_i$ for the final nucleus plus emergent particle and the initial nucleus plus projectile, respectively.

‡Kerman (62).

CROSS SECTIONS AS TRACES

$$\sigma_{\text{tot}} = \frac{1}{2S+1} \operatorname{tr} \hat{F}^{\dagger} \hat{F} \tag{2}$$

where the trace is performed with respect to all the variables,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2S+1} \operatorname{tr} \hat{F}^{\dagger} P_f \hat{F} P_i \tag{3}$$

where P_f and P_i are projection operators

$$P_f = |\hat{\mathbf{k}}_f\rangle\langle\hat{\mathbf{k}}_f|$$
$$P_i = |\hat{\mathbf{k}}_i\rangle\langle\hat{\mathbf{k}}_i|$$

so that, for example,

$$\langle\hat{\mathbf{k}}|P_f|\hat{\mathbf{K}}\rangle = \delta(\hat{\mathbf{k}} - \hat{\mathbf{k}}_f)\delta(\hat{\mathbf{k}}_f - \hat{\mathbf{K}}) \tag{4}$$

For orbital angular momentum states $|lm\rangle$,

$$\langle lm|P_f|l'm'\rangle = (-1)^{l-m} \sum_{LM} Y^*_{LM} \begin{pmatrix} l & L & l' \\ -m & M & m' \end{pmatrix} (l\|Y_L\|l') \tag{5}$$

Let

$$\hat{F} = \sum_{ll'J\mu} |l'SJM\rangle f_J(l',l) \langle lSJM| \tag{6}$$

where

$$|lSJM\rangle = \sum (lm; Sm_s|JM)|lm\rangle|Sm_s\rangle$$

and f_J also depends upon S. Then

$$\hat{F}(m_i \to m_f, \mathbf{k}_i \to \mathbf{k}_f) = \sum_{\substack{\lambda J \\ l,l' \\ \mu}} \mathscr{B}_{\lambda\mu}(l',l) i^{l'-l} (-1)^{J+m_f} f_{JS}(l',l) \begin{Bmatrix} l' & S & J \\ S & l & \lambda \end{Bmatrix} (Sm_i S - m_f|\lambda\mu) \tag{7}$$

where

$$\mathscr{B}_{\lambda\mu}(l',l) = \sqrt{\frac{2l+1}{4\pi}}\, Y_{l'\mu}(\hat{\mathbf{k}}_f)(l'\mu l0|\lambda\mu) \tag{8}$$

The z-axis has been taken along the incident projectile direction $\mathbf{k}_i$. The differential cross section is

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{2S+1} \sum_{\substack{L,J_1,J_2 \\ l_1 l'_1 \\ l_2 l'_2 \\ \mu}} (-1)^{J_1 - J_2} f_{J_1}(l'_1, l_1) f^*(l'_2, l_2) P_L(\cos\vartheta)$$

$$\times (l_1 S J_1 \| Y_L \| l_2 S J_2)(l'_2 S J_2 \| Y_L \| l'_1 S J_1) \tag{9}$$

POLARIZATION TENSORS: $T_{\lambda\mu}(\hat{S})$

$$T^\dagger_{\lambda\mu}(\hat{\mathbf{S}}) = (-1)^\mu T_{\lambda,-\mu}(\hat{\mathbf{S}}) \qquad \lambda \leqslant 2S \tag{10}$$

normalization

$$\langle S \| T_{\lambda\mu}(\hat{S}) \| S \rangle = \sqrt{2S+1} \tag{11}$$

$$\langle S\sigma | T_{\lambda\mu} | S\sigma' \rangle = (S\sigma'; \lambda\mu | S\sigma) \tag{12}$$

$$T_{00} = 1$$

$$T_{10} = \frac{S_z}{\sqrt{S(S+1)}} \qquad T_{1,\pm 1} = \frac{\mp (S_x \pm iS_y)}{\sqrt{2S(S+1)}} \tag{13}$$

$$T_{20} = \frac{3S_z^2 - S(S+1)}{\sqrt{S(S+1)(2S+3)(2S-1)}}$$

Expectation value of $T_{\lambda\mu}$ in the final state averaged over an unpolarized initial state,

$$\langle \overline{T_{\lambda\mu}} \rangle = \frac{1}{2S+1} \operatorname{tr} \hat{F}^\dagger P_f T_{\lambda\nu} \hat{F} P_i \tag{14}$$

$$= \frac{1}{(2S+1)\sqrt{4\pi}} \sum (-1)^{J_2 - J_1} Y^*_{L'M'}(\hat{\mathbf{k}}_f) \frac{(L'M'; \lambda, \mu | L0)}{\sqrt{2L+1}}$$

$$\times (l_1 S J_1 \| Y_L \| l_2 S J_2)(l'_2 S J_2 \| T_\Lambda(L', \lambda) \| l'_1 S J_1)$$

$$\times f_{J_1}(l'_1, l_1) f^*_{J_2}(l'_2, l_2) \tag{15}$$

where

$$T_\Lambda(L', \lambda) = (Y_{L'} \otimes T_\lambda)^\Lambda$$

$$T_{\Lambda,M} = \sum_{q,M'} (L' - M'; \lambda q | \Lambda, M) Y_{L' - M'} T_{\lambda q} \tag{16}$$

SPIN-$\frac{1}{2}$ PROJECTILE

Let the polarization induced by a scatterer on an unpolarized beam be

$$\mathbf{P}_s = P_s \mathbf{n}_s$$

where $\mathbf{n}_s$ is a unit vector normal to the scattering plane:

$$\mathbf{n}_s = \frac{(\mathbf{k}_i \times \mathbf{k}_f)}{|\mathbf{k}_i \times \mathbf{k}_f|} \tag{17}$$

where $\mathbf{k}_i$ is the incident direction and $\mathbf{k}_f$ the final direction of the spin-$\frac{1}{2}$ projectile.

The cross section for the scattering of a beam with polarization $P\mathbf{n}$ by the scatterer above is

$$\sigma_{0s} = \sigma_0(1 + PP_s \mathbf{n}\cdot\mathbf{n}_s) \tag{18}$$

where σ_0 is the cross section for an unpolarized beam.

The general expression for the scattering of a polarized beam with polarization $\mathbf{P}$ is

$$\begin{aligned}\sigma_s \mathbf{P}_s = \sigma_{0s}\{&\mathbf{n}[P_{0s} + D_s \mathbf{n}\cdot\mathbf{P}] + (\mathbf{n}\times\hat{\mathbf{k}}_f)[A_s \hat{\mathbf{k}}_i\cdot\mathbf{P} + R_s(\mathbf{n}\times\hat{\mathbf{k}}_i)\cdot\mathbf{P}] \\ &+ \hat{\mathbf{k}}_f[A_s'(\hat{\mathbf{k}}_i\cdot\mathbf{P}) + \mathbf{R}_s'(\mathbf{n}\times\hat{\mathbf{k}}_i)\cdot\mathbf{P}]\}\end{aligned} \tag{19}$$

TIME-REVERSAL INVARIANCE

$$A = -R' \tag{20}$$

If the incident beam is unpolarized,

$$\mathbf{P}_s = \mathbf{n}P_{0s} \tag{21}$$

If the polarization of the incident beam is in the $\mathbf{n}$ direction,

$$\mathbf{P}_s = \mathbf{n}(P_{0s} + PD_s) \tag{22}$$

D_s is the depolarization. Under time-reversal invariance, $-1 + 2P_s \leqslant D_s \leqslant 1$:

$$D_s = 1 \qquad \text{spin 0 target} \tag{23}$$

$$Q_s = -[A_s \cos\vartheta + R_s \sin\vartheta] = \frac{2\,\mathrm{Im}\,A^*B}{|A|^2 + |B|^2} \qquad \text{spin 0 target} \tag{24}$$

$$Q'_s = R_s \cos\vartheta - A_s \sin\vartheta = |A|^2 - |B|^2 \qquad \text{spin 0 target} \tag{25}$$

$$P_s = \frac{2\,\mathrm{Re}\,A^*B}{|A|^2 + |B|^2} \qquad \text{spin 0 target} \tag{26}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathbf{unpol}} = |A|^2 + |B|^2 \qquad \text{spin 0 target} \tag{27}$$