Lecture 2
Evolution and resummation

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Outlines

- Resummation technique
- k_T resummation
- BFKL equation
- Threshold resummation
- DGLAP equation
- Joint resummation

Introduction

- Factorization introduces factorization scale into hard kernel and PDF

$$F(Q^2) = H(Q, \mu_f) \otimes \phi(Q_0, \mu_f)$$

- Higher-order corrections produce

$$\ln(Q/\mu_f),\ \ln(\mu_f/Q_0)$$

- due to splitting of log in F

$$\ln(Q/Q_0) = \ln(Q/\mu_f) + \ln(\mu_f/Q_0)$$

- Set $\mu_f = Q$ to eliminate log in H
- Need PDF inputs $\phi(Q_0, Q)$ for arbitrary Q?

RG equations

- Factorization scale does not exist in QCD diagrams, but is introduced when physical quantity factorized into H and PDF
- Renormalization-group equations

$$\mu_f \frac{d}{d\mu_f} F(Q^2) = 0$$

$$\mu_f \frac{d}{d\mu_f} \phi(Q_0, \mu_f) = \gamma_\phi \phi(Q_0, \mu_f)$$

$$\mu_f \frac{d}{d\mu_f} H(Q, \mu_f) = -\gamma_\phi H(Q, \mu_f)$$

- γ_ϕ is anomalous dimension of PDF

RG evolution

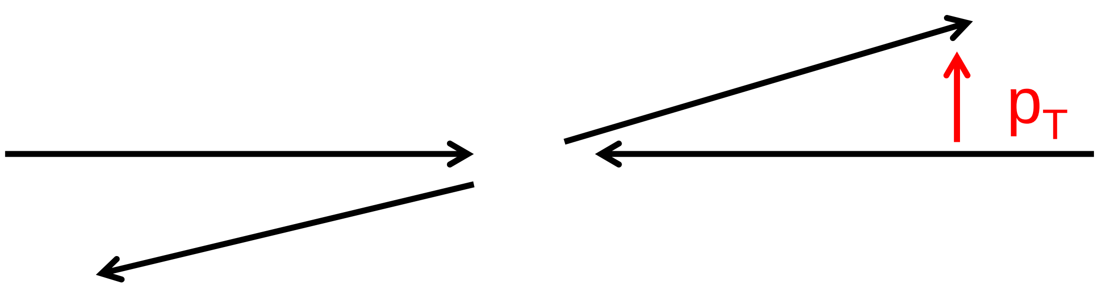
- Solution of RG equations describes Q evolution of PDF

$$\phi(Q_0,Q)=\phi(Q_0,Q_0)\exp\left(\int_{Q_0}^{Q}\frac{d\mu_f}{\mu_f}\gamma_\phi(\mu_f)\right)$$

- Evolution from summation of $\ln(Q/Q_0)$
- just need to extract PDF at $Q_{0,}\ \phi(Q_0,Q_0)$
- PDF at other Q is known via evolution
- Evolution increases predictive power of factorization theorem

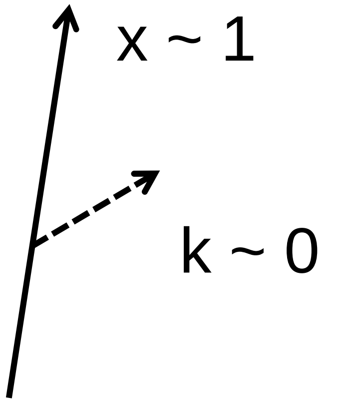
Extreme kinematics

- QCD processes in extreme kinematic region, such as low p_T and large x, generate double logs
- Limited phase space for real corrections
- Low p_T jet, photon, W,… requires small p_T real gluon emissions
- Top pair production requires large x



Double logs

- Large x demands soft real gluon emission
- Cancellation between virtual and real corrections is not exact



- Large double logs are produced,

$$\alpha_S \ln^2(E/p_T), \alpha_S \ln(1-x)/(1-x)$$

E being beam energy

- Sum logs to all orders---resummation
- Resummation improves perturbation

Evolution and resummation

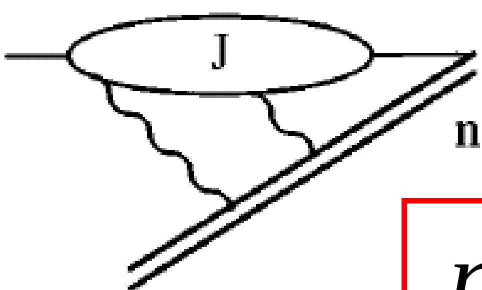
- RG evolution (UV dynamics) can be found in standard textbook
- Cover evolution equations involving splitting kernels (IR dynamics): DGLAP equation for PDF (collinear factorization), BFKL equation for TMD (k_T factorization)
- Cover threshold resummation for large x (collinear factorization) and k_T resummation for small k_T (k_T factorization)

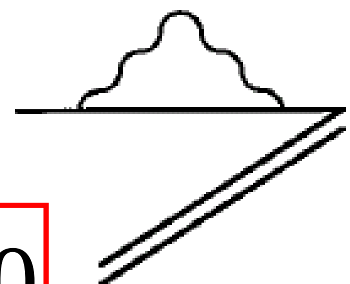
Resummation technique

Jet function in covariant gauge

- Quark jet function at amplitude level

$$J(p,n)=\left\langle 0\left|P\exp\left[ig\int_0^\infty dz\; n\cdot A(nz)\right]q(0)\right|p\right\rangle$$





$$\boxed{n^2 \neq 0}$$

- NLO diagrams contain double logarithms
- Perform resummation in covariant gauge

Key idea

- Derive differential equation $p^+ dJ/dp^+ = CJ$
- C contains only single logarithm
- Treat C by RG equation
- Solve differential equation, and solution resums double logarithms
- J depends on Lorentz invariants p.n, n^2
- Feynman rules for Wilson lines show scale invariance in n, $n_\nu / n \cdot l$
- J must depend on $(p \cdot n)^2 / n^2$

Derivative with respect to n

- Derivative with respect to p can be replaced by derivative with respect to n

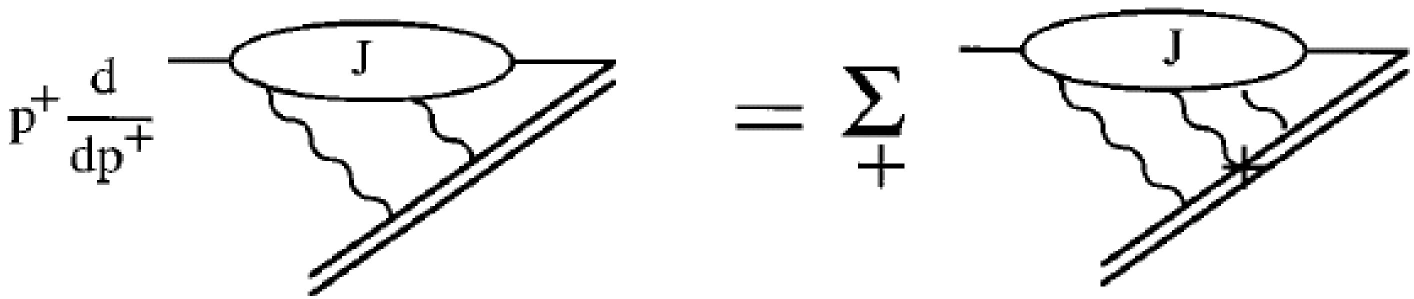
$$p^{+} \frac{d}{dp^{+}} J = -\frac{n^{2}}{v \cdot n} v_{\alpha} \frac{d}{dn_{\alpha}} J$$

- Collinear dynamics independent of n
- Variation does not contain collinear dynamics, the reason why C contains only single logarithm
- Variation effect can be factorized

Special vertex

- n appears only in Wilson lines

$$-\frac{n^2}{v\cdot n}\, v_\alpha \frac{d}{dn_\alpha}\frac{n_\mu}{n\cdot l} = \frac{n^2}{v\cdot n}\left(\frac{v\cdot l}{n\cdot l}\, n_\mu - v_\mu\right)\frac{1}{n\cdot l} \equiv \frac{\hat{n}_\mu}{n\cdot l}$$



- If gluon momentum l parallel to p, v.l vanishes.
- Contraction of v with J, dominated by collinear dynamics, also vanishes

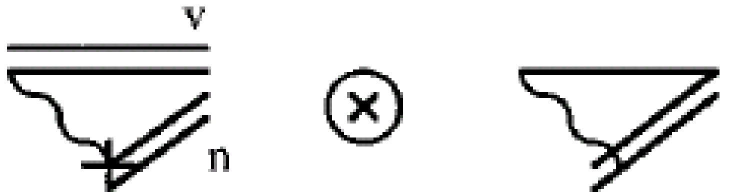
Soft factorization

- Two-loop diagrams as example

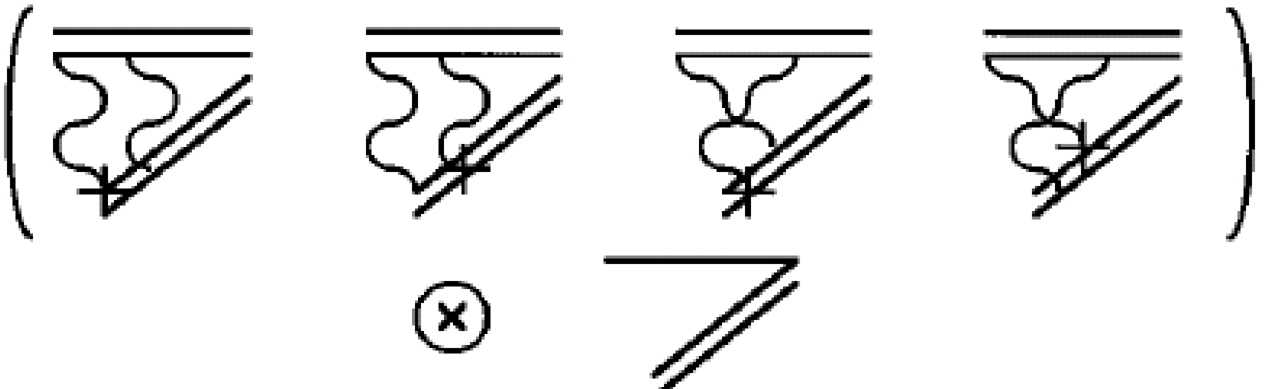
- If l flowing through special vertex is soft, but another is not, only 1st diagram dominates
- Another gluon with finite momentum smears soft divergence in other three diagrams

Soft kernel

- Factorizing the soft gluon with eikonal approximation



- If another gluon is also soft, we get two-loop soft kernel K



Hard factorization

- If l flowing through special vertex is hard, only 2nd diagram dominates,
- It suppresses others, as another gluon is not hard
- Factorize hard kernel with Fierz transformation

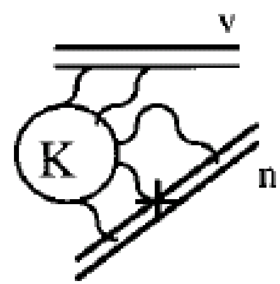
- If both gluons are hard, they contribute to two-loop hard kernel

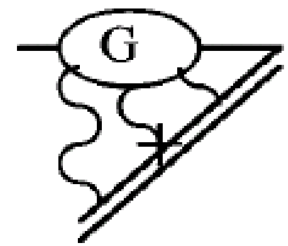
Differential equation

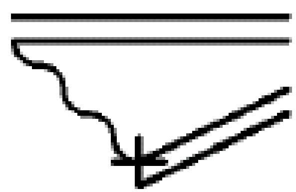
- Extending the factorization to all orders

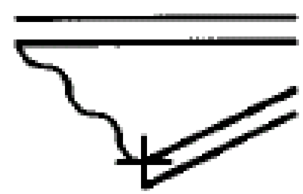
$$p^+ \frac{d}{dp^+} J = [K(m/\mu, \alpha_s(\mu)) + G(p^+ \nu/\mu, \alpha_s(\mu))]J$$

- Kernels K and G are described by





- At LO, K = 

- G =  − 

Integrals for kernels

$$K = -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{\hat{n}_\mu}{n\cdot l} \frac{g^{\mu\nu}}{l^2 - m^2} \frac{v_\nu}{v\cdot l} - \delta K$$

$$= -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{n^2}{(n\cdot l)^2 (l^2 - m^2)} - \delta K$$

$$G = -ig^2 C_F \mu^\epsilon \int \frac{d^{4-\epsilon}l}{(2\pi)^{4-\epsilon}} \frac{\hat{n}_\mu}{n\cdot l} \frac{g^{\mu\nu}}{l^2} \left(\frac{\not{p} + \not{l}}{(p+l)^2} \gamma_\nu - \frac{v_\nu}{v\cdot l} \right)$$

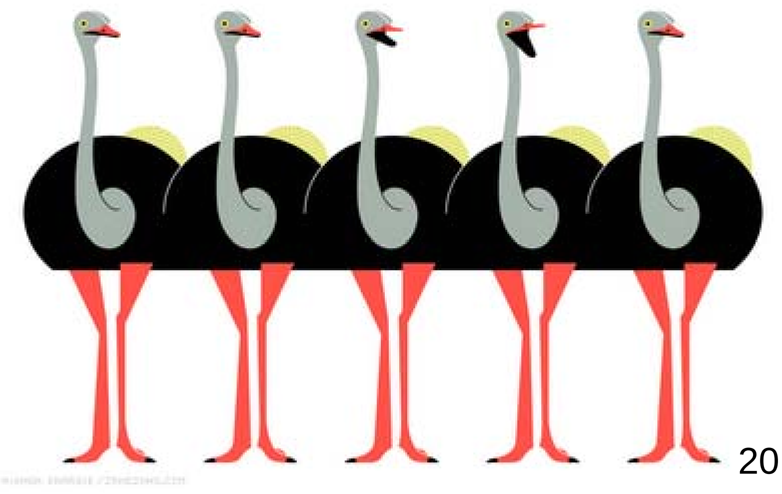
$$- \delta G,$$

- δK , δG additive counterterm

- Chinese saying: One stone, two birds

Goals of this lecture

- Now one stone, five birds:
- Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation
- Threshold resummation
- Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation
- k_T resummation
- Joint resummation



Master equation for TMD

- Consider TMD

$$\phi(x,k_T,p^+) = \int \frac{dy^-}{2\pi}\int \frac{d^2y_T}{4\pi^2} e^{-ixp^+y^- + i\mathbf{k}_T\cdot\mathbf{y}_T}\langle p|\bar{q}(y^-,\mathbf{y}_T)\frac{1}{2}\gamma^+ q(0)|p\rangle$$

- Three scales: $(1-x)p^+ \quad xp^+ \quad k_T.$

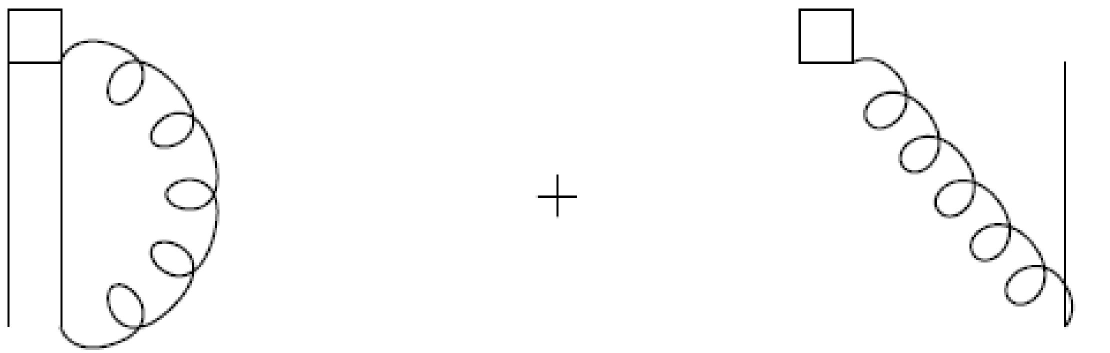
ϕ depends on p^+ via the ratio $(p\cdot n)^2/n^2$

- Derivative

$$p^+\frac{d}{dp^+}\phi = -\frac{n^2}{v\cdot n}v_\alpha\frac{d}{dn_\alpha}\phi$$

Kernels (in axial gauge)

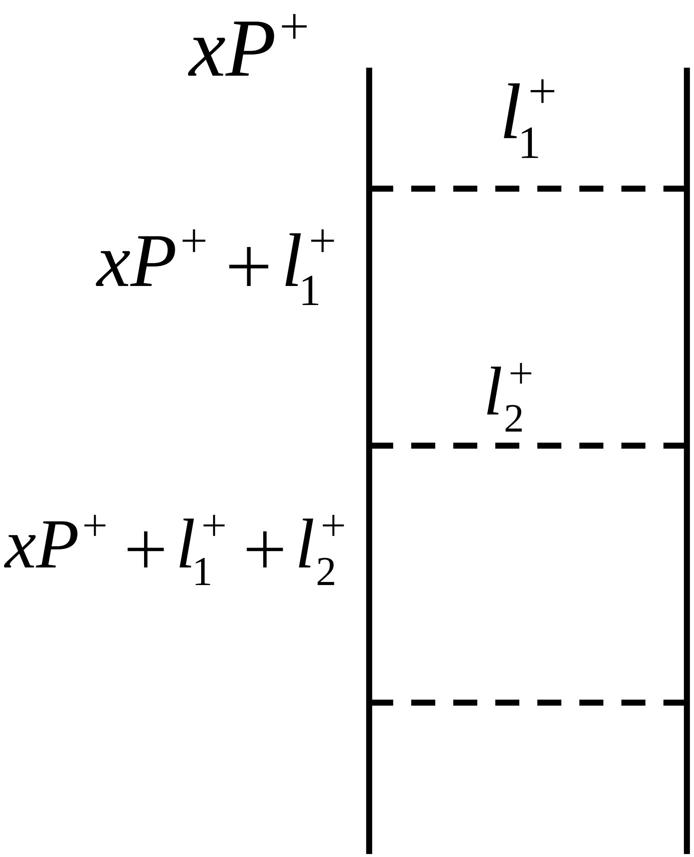
- Soft kernel K (virtual + real)



- Hard kernel G (virtual – soft subtraction)

k_T resummation and BFKL equation

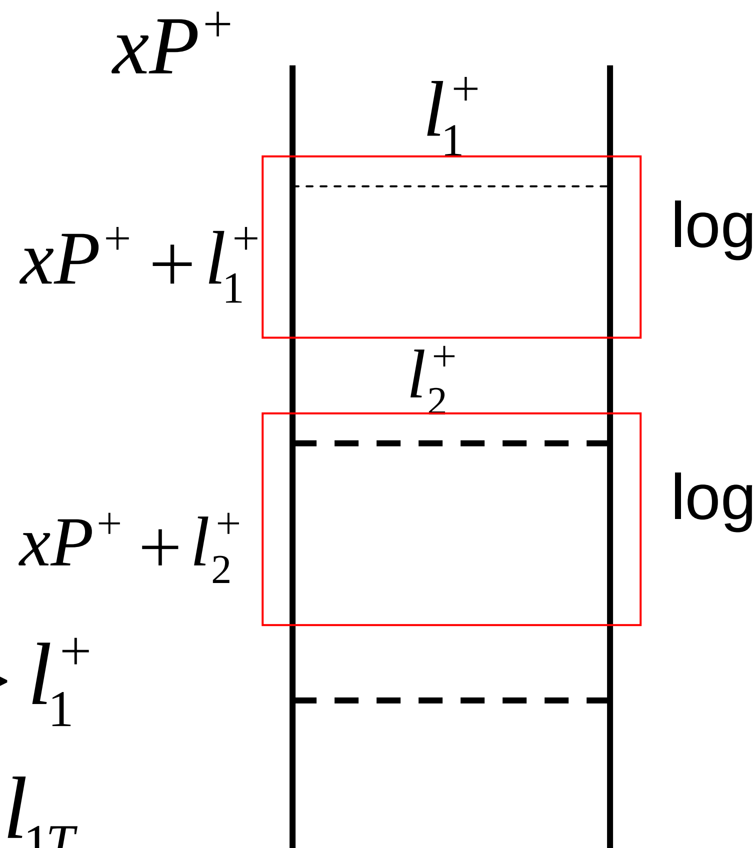
Rapidity ordering



$$\approx$$

$$l_2^+ >> l_1^+$$

$$l_{2T} \approx l_{1T}$$



Approximation under rapidity ordering

- TMD is independent of l+ under rapidity ordering. Soft approximation

$$\phi(x+l^+/p^+,|\mathbf{k_T}+\mathbf{l_T}|)\approx\phi(x,|\mathbf{k_T}+\mathbf{l_T}|)$$

- k_T and l_T are of the same order
- l+ integrated up to infinity, the scale $(1-x)p^+$ does not exist
- Usually work in conjugate space in b via Fourier transformation

k_T resummation

- Two scales xp^+ k_T.
- Solution of resummation equation sums double logs of b

$$\phi(x,b,p^+) = \Delta_k(b,xp^+)\phi^{(0)}(x)$$

- Sudakov exponential

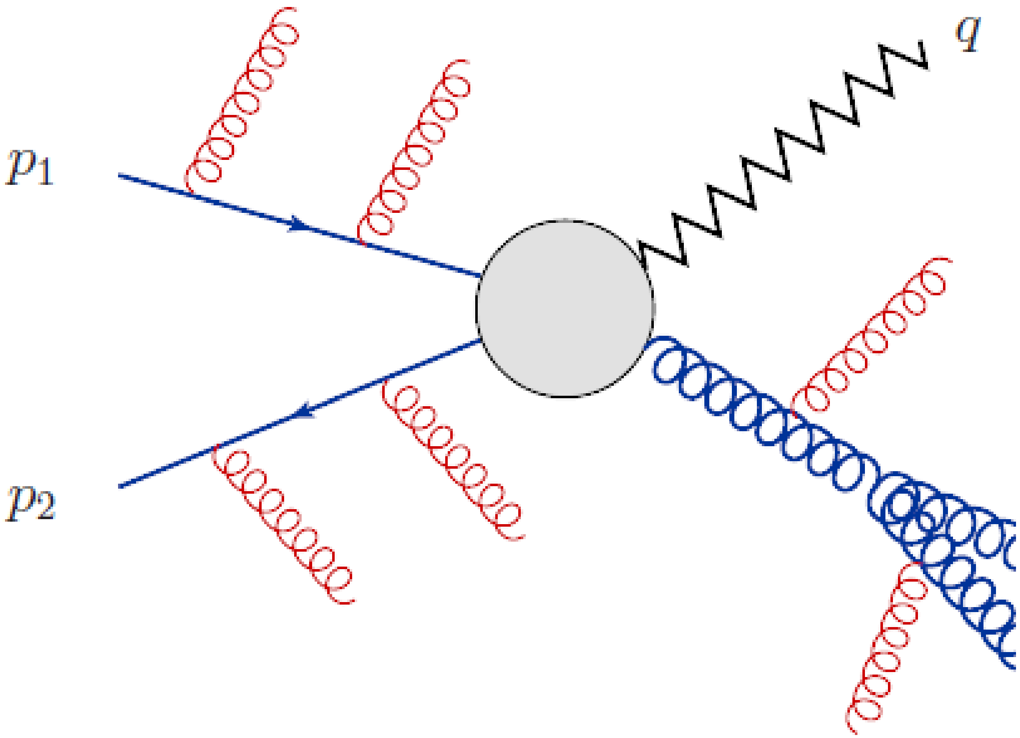
$$\Delta_k(b,xp^+) = \exp\left[-2\int_{1/b}^{xp^+}\frac{dp}{p}\int_{1/b}^{p}\frac{d\mu}{\mu}\gamma_K(\alpha_s(\mu))\right]$$

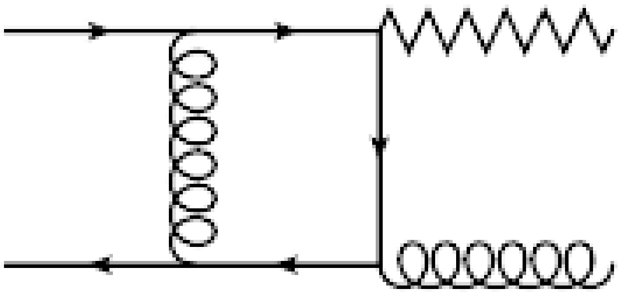
- Anomalous dimension

$$\gamma_K = \frac{\alpha_s}{\pi}C_F + \left(\frac{\alpha_s}{\pi}\right)^2 C_F\left[C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-\frac{5}{18}n_f\right]$$

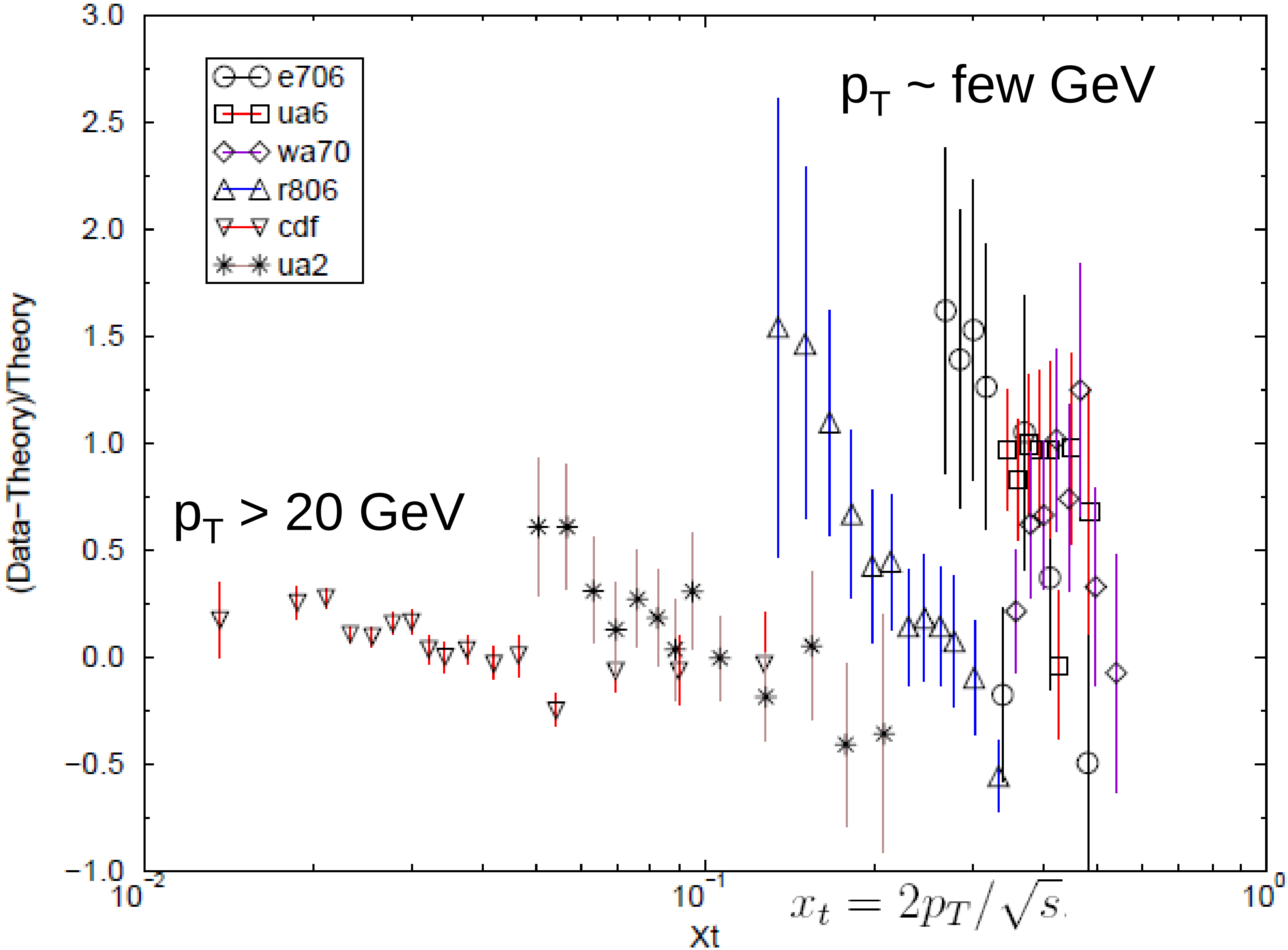
Direct photon

- Soft and collinear radiation at low p_T

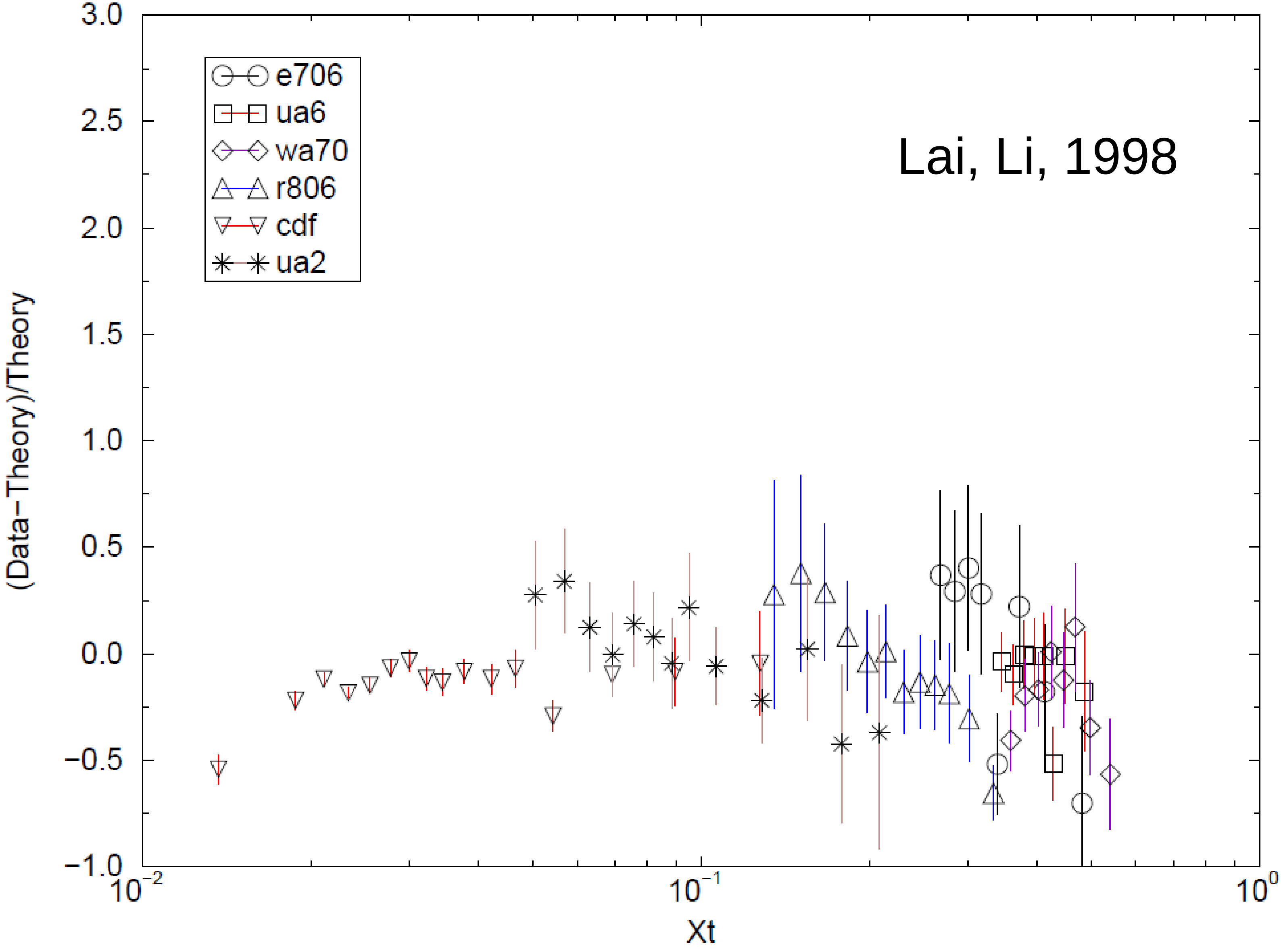




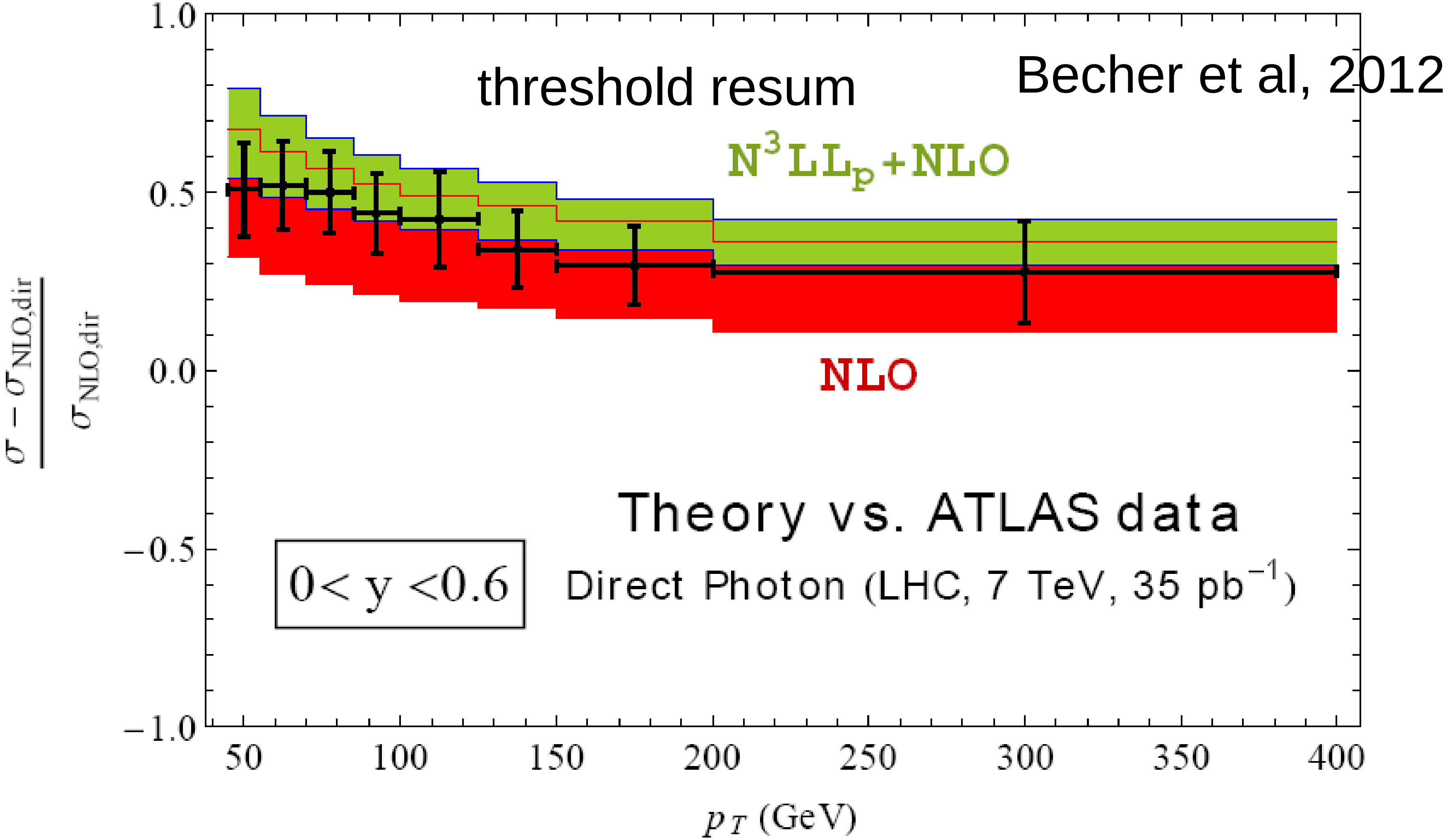
Direct photon before resum



Direct photon after k_T resum



High p_T data of direct photon



k_T resum for low p_T, threshold resum for high p_T?

Small x

- At small x with $xp^+ \sim k_T$

$$p^+ \frac{d}{dp^+}\phi(x, k_T, p^+) = -x\frac{d}{dx}\phi(x, k_T, p^+)$$

- Two scale system reduces to single-scale
- Consider gluon TMD, $C_F = 4/3 \;\rightarrow\; N_c = 3$

$$\begin{aligned} F(x, k_T) &= \frac{1}{p^+}\int \frac{dy^-}{2\pi}\int \frac{d^2 y_T}{4\pi} e^{-i(xp^+ y^- - \mathbf{k}_T \cdot \mathbf{y}_T)} \\ &\quad \times \frac{1}{2}\sum_\sigma \langle p, \sigma | F_\mu^{+}(y^-, y_T) F^{\mu +}(0) | p, \sigma \rangle \end{aligned}$$

BFKL equation

- Only soft scale exists. Drop G
- Keep soft kernel with a UV cutoff

$$\bar{\phi}(x, k_T, p^+) = ig^2 N_c \int \frac{d^4 l}{(2\pi)^4} N_{\nu\beta}(l) \frac{\hat{v}^\beta v^\nu}{v \cdot l} \left[\frac{\theta(k_T^2 - l_T^2)}{l^2} \phi(x, k_T, p^+) \right.$$
$$\left. + 2\pi i \delta(l^2) \phi(x, |\mathbf{k}_T + \mathbf{l}_T|, p^+) \right] ,$$

- BFKL equation

$$\bar{\alpha}_s = N_c \alpha_s / \pi$$

$$\frac{d\phi(x, k_T, p^+)}{d \ln(1/x)} = \bar{\alpha}_s \int \frac{d^2 l_T}{\pi l_T^2} \left[\phi(x, |\mathbf{k}_T + \mathbf{l}_T|, p^+) - \theta(k_T^2 - l_T^2) \phi(x, k_T, p^+) \right]$$

Leading-log BFKL

- High-energy cross section predicted by LL BFKL

$$\hat{\sigma} \approx \frac{1}{\hat{t}} \left(\frac{\hat{s}}{|\hat{t}|} \right)^{\omega_P - 1}$$

momentum transfer squared

center-of-mass energy-squared

- Pomeron intercept ω_P

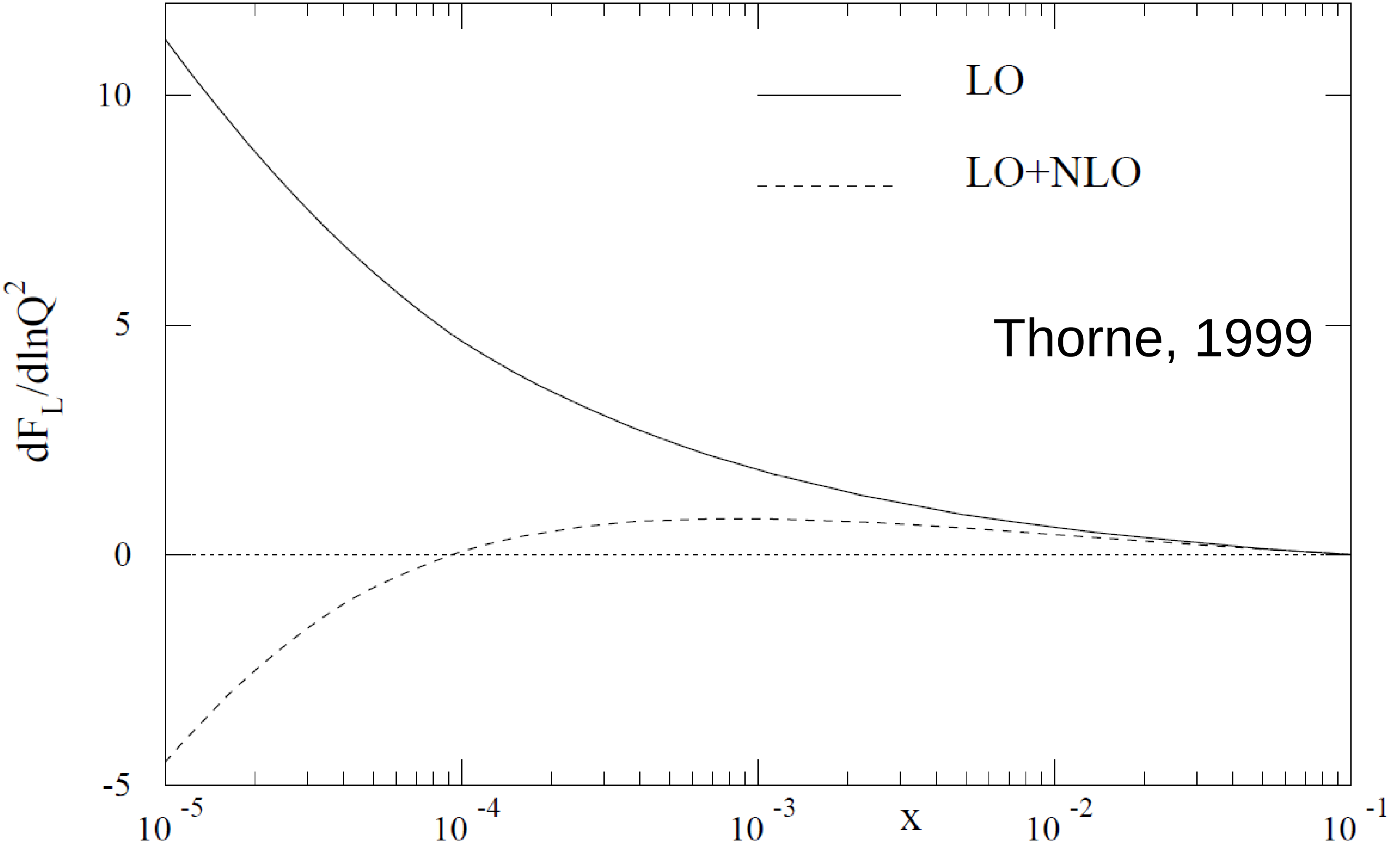
$$\omega_P - 1 = \frac{4 N_c \alpha_s \ln 2}{\pi}$$

- Violate Froissart (unitarity) bound

$$\sigma_{\text{tot}} \leq \text{const.} \times \ln^2 s$$

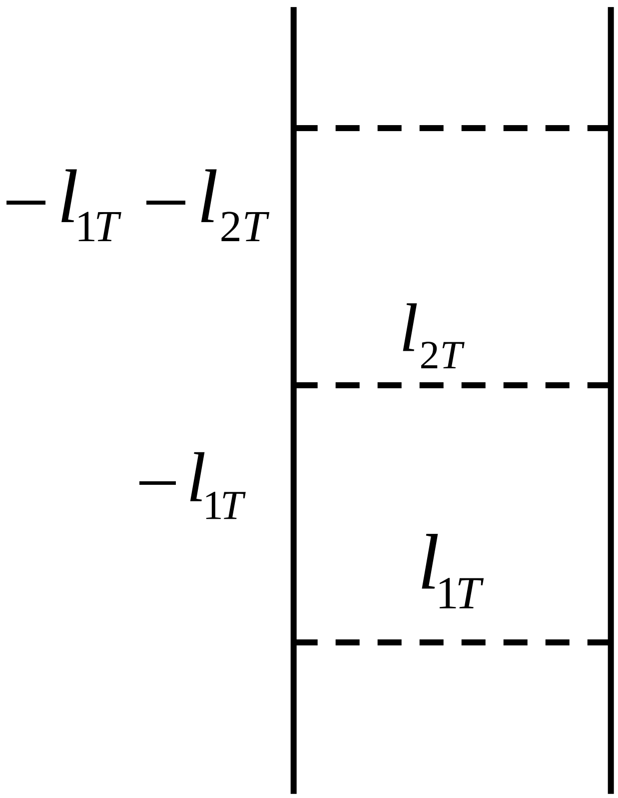
NLL BFKL

- Unsatisfactory prediction of LL BFKL called for NLL corrections---but too big



Threshold resummation and DGLAP equation

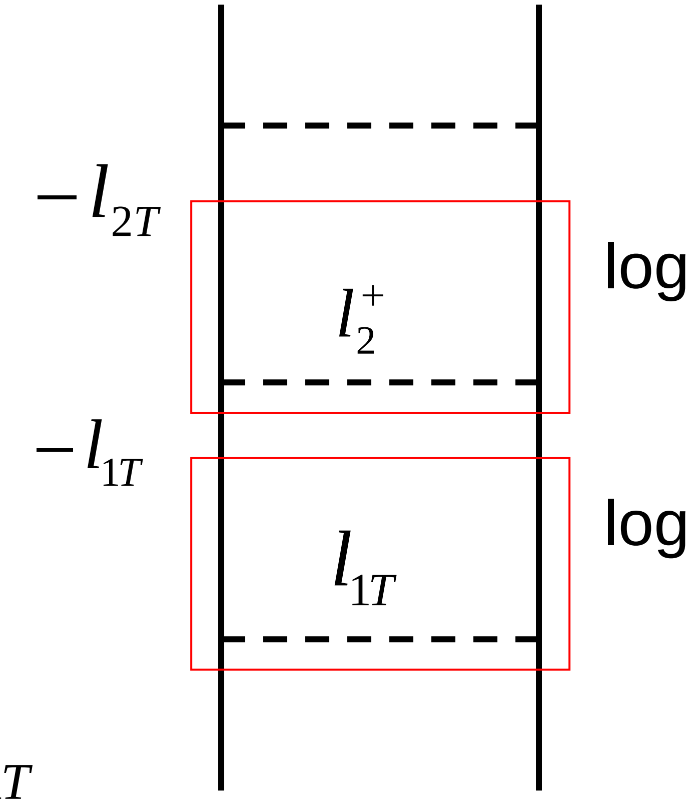
k_T ordering



$$\approx$$

$$l_2^+ \approx l_1^+$$

$$l_{2T} >> l_{1T}$$



Approximation under k_T ordering

- TMD is independent of l_T under k_T ordering
- Soft approximation

$$\phi(x + l^+/p^+, |\mathbf{k_T} + \mathbf{l_T}|) \approx \phi(x + l^+/p^+, k_T)$$

- x and l+/p+ are of the same order
- Integrate over k_T, TMD -> PDF
- l_T integrated up to infinity, the scale k_T disappears

Threshold resummation

- At large x, two scales $(1-x)p^+ \quad xp^+$
- Mellin transformation

$$\bar{\phi}_{sr}(N,p^+) \;=\; \int_0^1 dx x^{N-1} \bar{\phi}_{sr}(x,p^+)$$

$$xp^+ \sim p^+ \quad (1-x)p^+ \sim p^+/N$$

- Rewrite $$p^+ \frac{d\phi}{dp^+} = \frac{p^+}{N} \frac{\partial \phi}{\partial (p^+/N)}$$

Solution

- Solution of resummation equation sums double logs of N

$$\phi(N,p^+) = \Delta_t(N,p^+)\phi^{(0)}$$

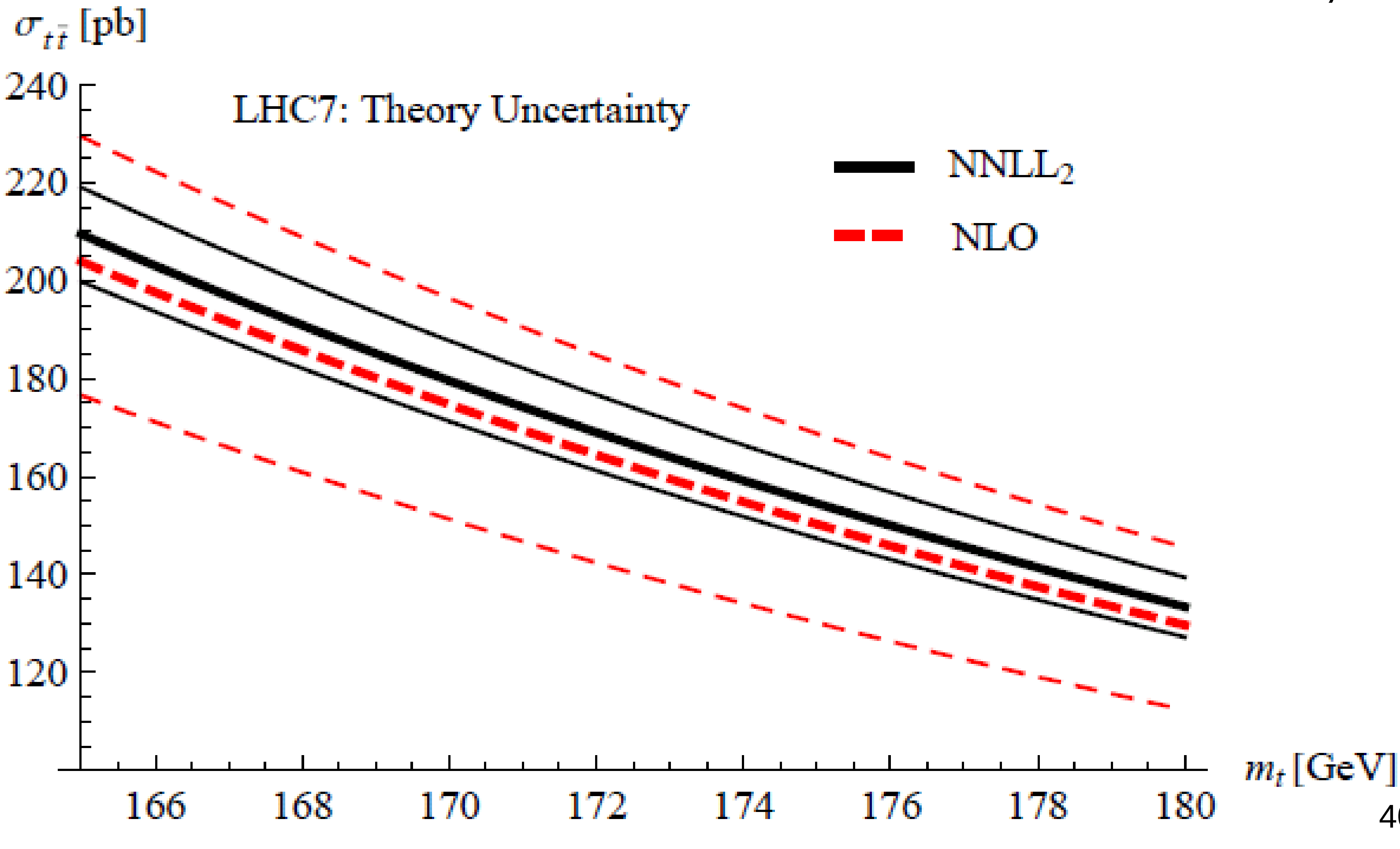
$$\Delta_t(N,p^+) = \exp\left[-2\int_{p^+/N}^{p^+}\frac{dp}{p}\int_p^{p^+}\frac{d\mu}{\mu}\gamma_K(\alpha_s(\mu))\right]$$

- Equivalent expression

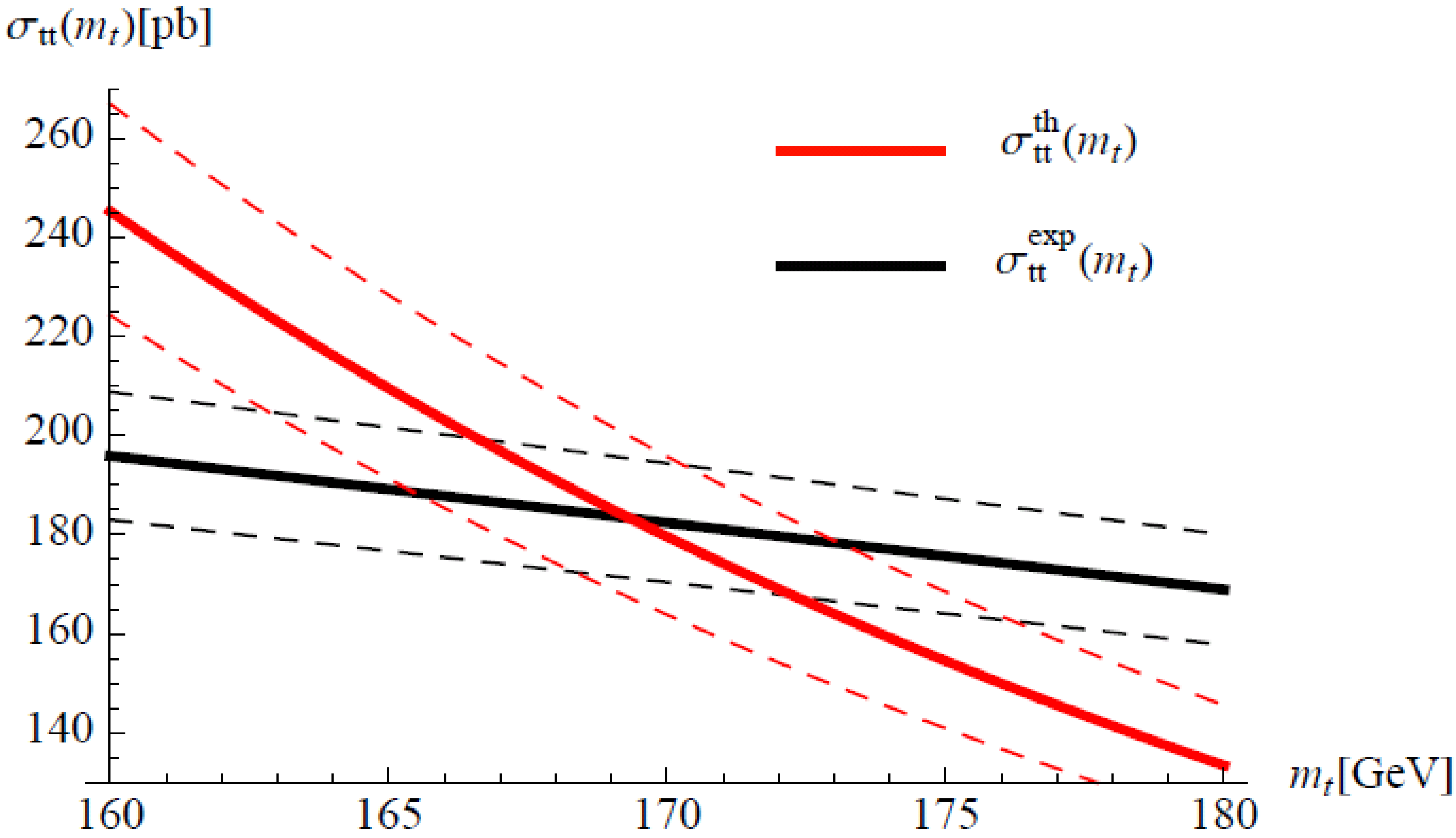
$$\Delta_t(N,p^+) = \exp\left[-\int_0^1 dz\frac{z^{N-1}-1}{1-z}\int_{(1-z)^2}^1\frac{d\lambda}{\lambda}\gamma_K(\alpha_s(\sqrt{\lambda}p^+))\right]$$

Top-quark pair production with NNLL threshold resummation

Beneke et al., 2011



Determination of top mass



DGLAP equation

- Intermediate x, $xp^+ \sim (1-x)p^+$
- Single scale
- DGLAP equation

$$p^+ \frac{d}{dp^+}\phi(x,p^+) = \int_x^1 \frac{d\xi}{\xi} P(x/\xi, p^+)\phi(\xi, p^+)$$

$$P(z,p^+) = \frac{\alpha_s(p^+)}{\pi} C_F \frac{2}{(1-z)_+}$$

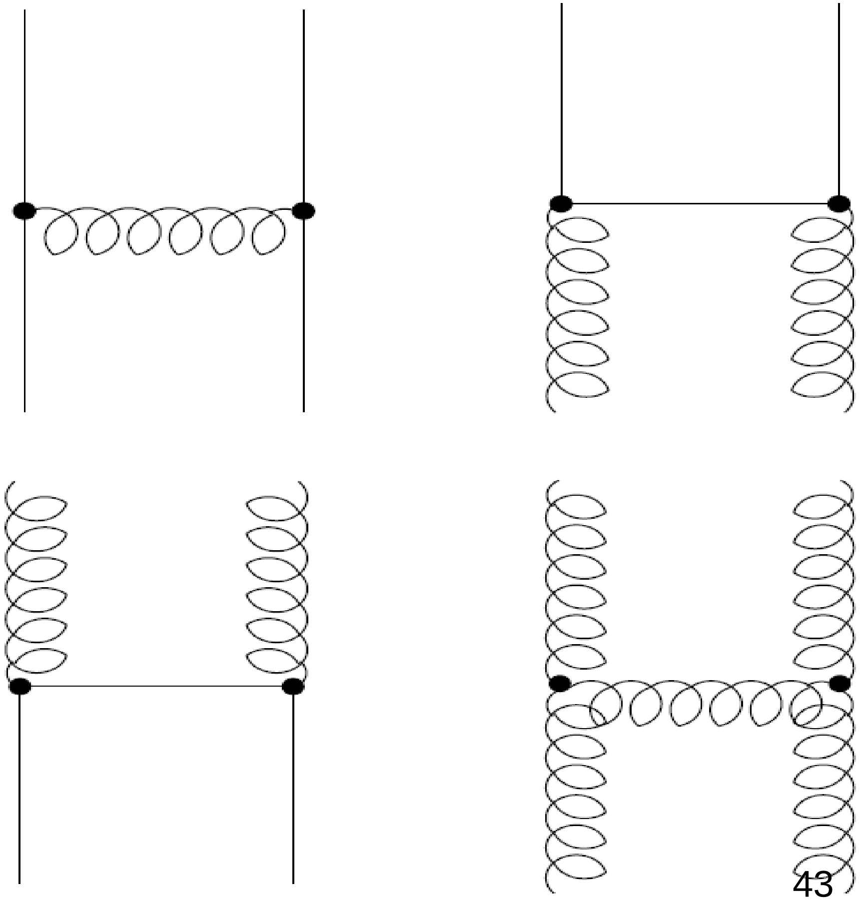
- z -> 1 limit of splitting kernel

$$P_{qq} = (\alpha_s C_F/\pi)(1+z^2)/(1-z)_+$$

Complete DGLAP kernels

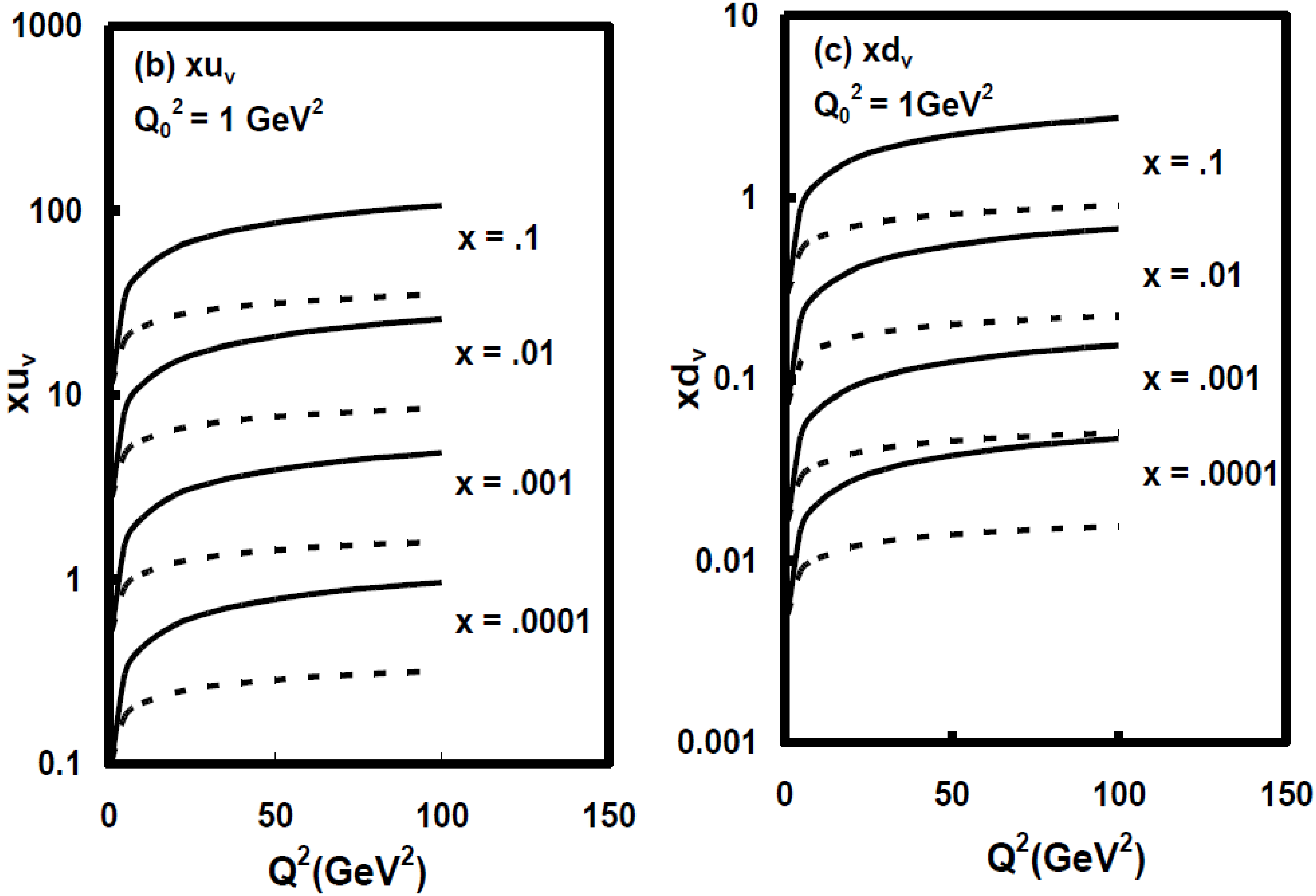
$$\frac{\partial}{\partial \ln Q^2} \begin{pmatrix} q_S \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \bigotimes \begin{pmatrix} q_S \\ g \end{pmatrix}$$

$$a(x) \bigotimes b(x) \equiv \int_x^1 \frac{d\omega}{\omega} a(\omega) b(\frac{x}{\omega})$$

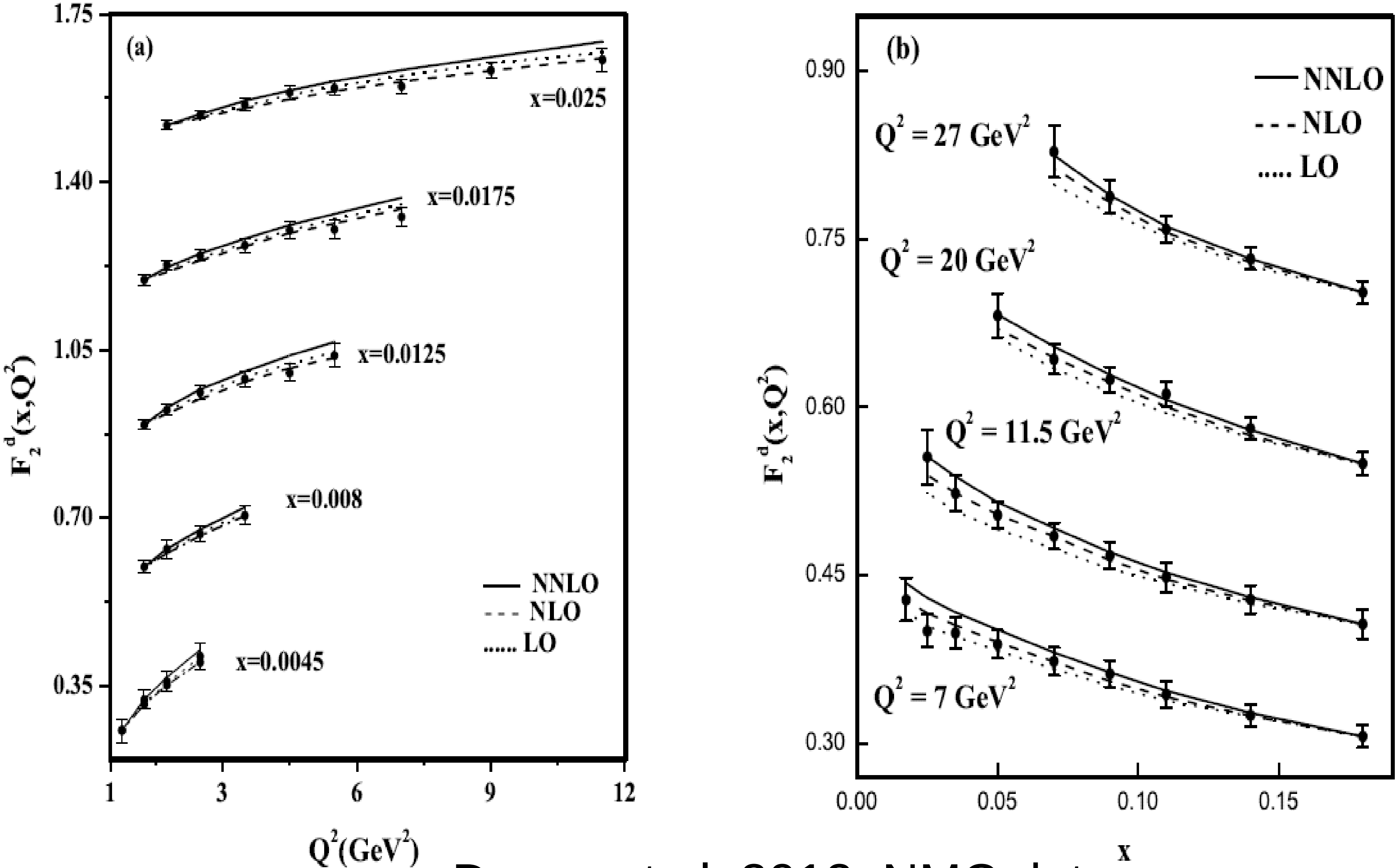


splitting kernels have been calculated to two loops

Valence quark LO evolution



Deuteron structure function



Devee et al, 2012, NMC data

Joint resummation

Li, 1999;
Laenen et al., 2000

- k_T resummation

$$\phi(x,b,p^+) = \exp\left[-2\int_{1/b}^{xp^+}\frac{dp}{p}\int_{1/b}^{p}\frac{d\mu}{\mu}\gamma_K(\alpha_s(\mu))\right]\phi^{(0)}$$

- Threshold resummation

$$\phi(N,p^+) = \exp\left[-2\int_0^1 dz\frac{z^{N-1}-1}{1-z}\int_{1-z}^{1}\frac{d\lambda}{\lambda}\gamma_K(\alpha_s(\lambda p^+))\right]\phi^{(0)}$$

- Same anomalous dimension
- Unification of the two resummations?

General soft approximation

- Keep both l+ and l_T dependence

$$\phi(x + l^+/p^+, |\mathbf{k}_T + \mathbf{l}_T|, p^+)$$

- Joint resummation

$$\phi(N, b, p^+) = \exp\left[-2\int^{p^+} \frac{dp}{p}\int^{p} \frac{d\mu}{\mu}\gamma_K(\alpha_s(\mu))\right]\phi^{(0)}$$

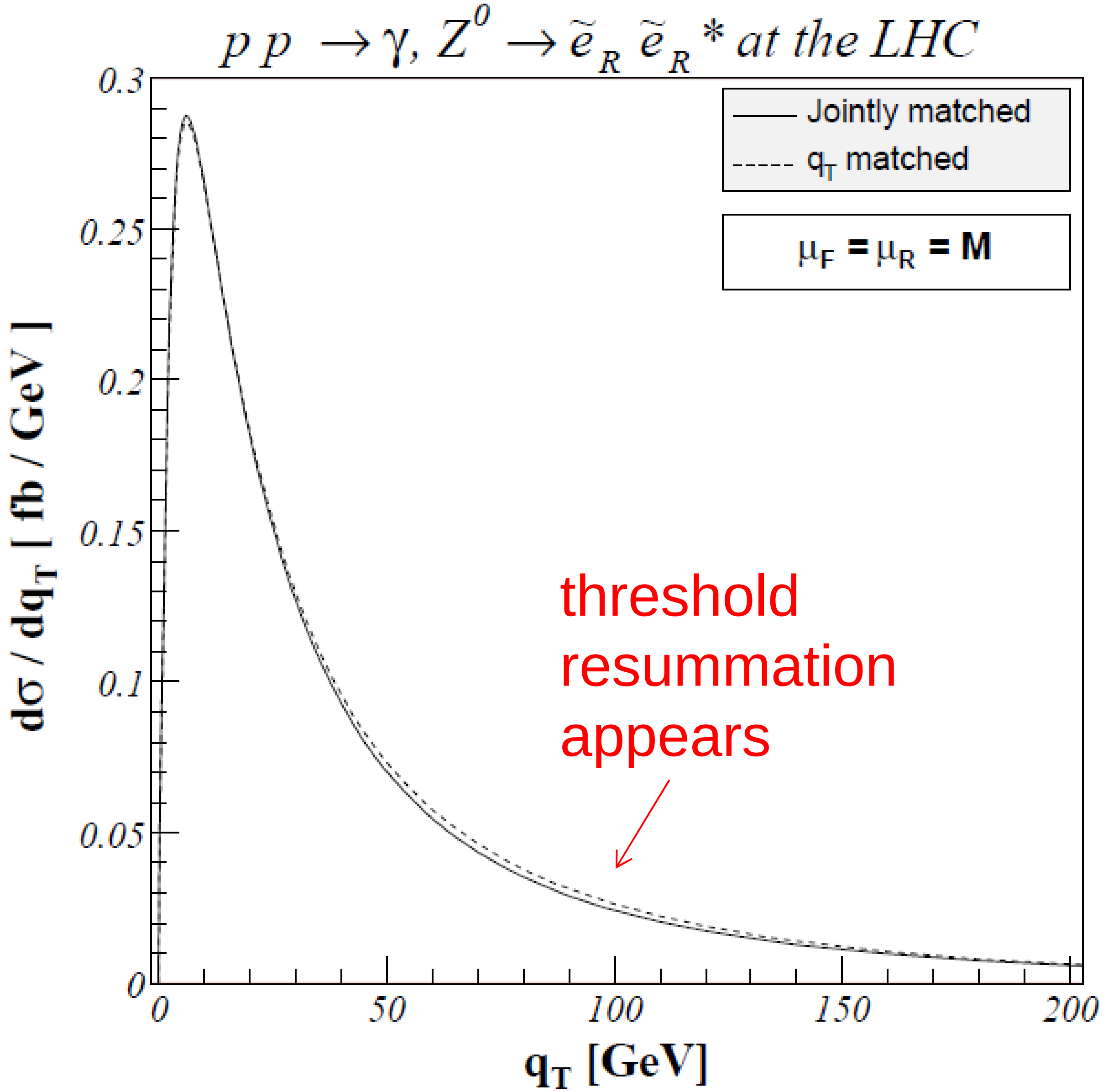
$$\chi(\bar{N}, \bar{b}) = \frac{\bar{N}}{1 + \eta\,\bar{b}/\bar{N}} + \bar{b}$$

$$\bar{N} = N e^{\gamma_E}$$

$$\bar{b} = bMe^{\gamma_E}/2$$

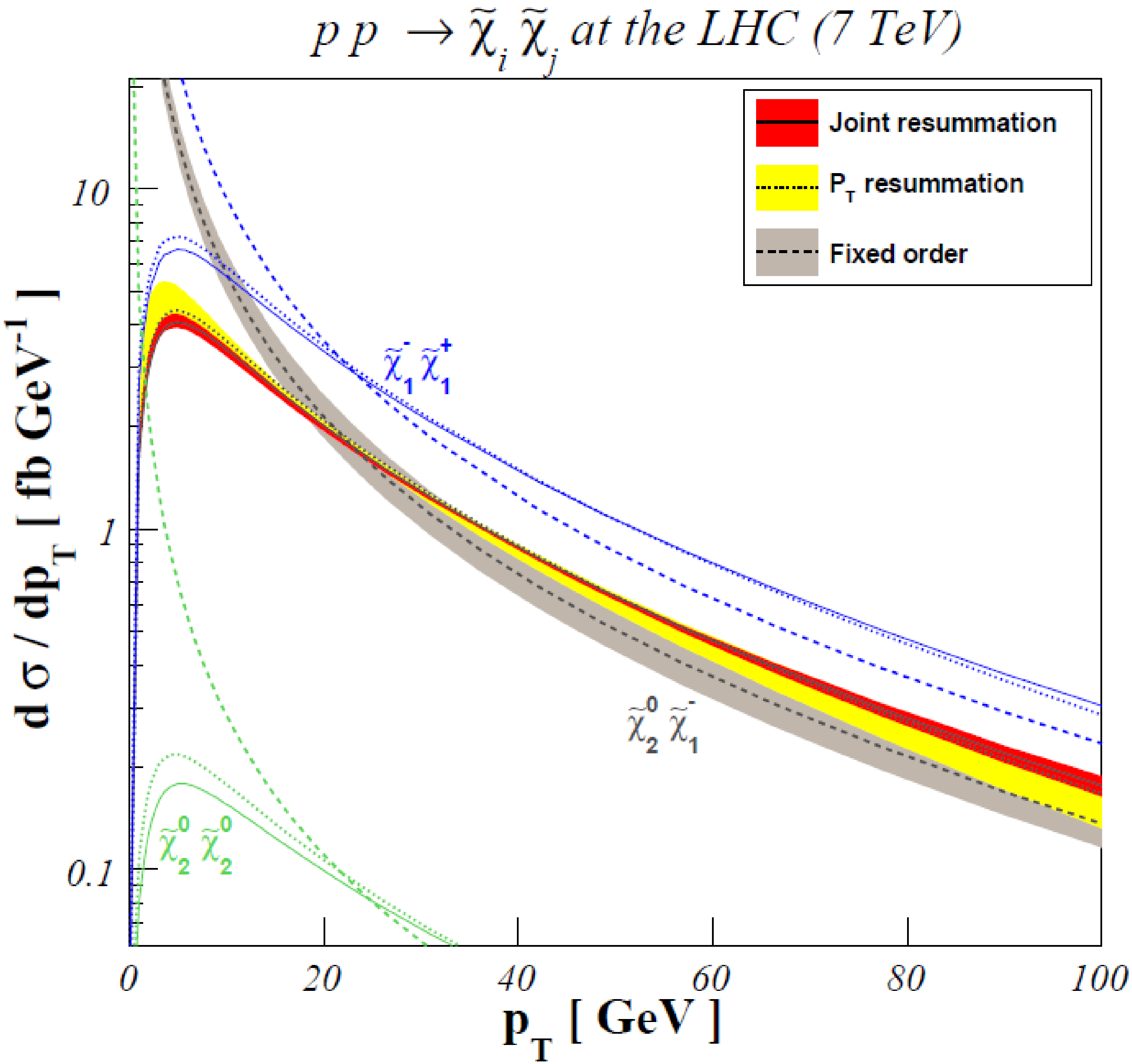
- Large b, joint $\longrightarrow$ k_T resummation
- Large N, joint $\longrightarrow$ threshold resummation

Resummation effect



Fuks, 2007

Resummation effect



Debove et al, 2011

Summary

- Sophiscated evolution and resummation techniques have been developed in PQCD
- Predictive power enhanced
- Perturbation improved
- Precision increased
- Resummation of other logs need to be developed: rapidity logs, non-global logs,…