Quantum restoration of broken symmetries

V.V. Belokurov[1,2] and E.T. Shavgulidze[1]

1. Lomonosov Moscow State University, Russia
2. Institute for Nuclear Research
of Russian Academy of Sciences, Russia
vvbelokurov@yandex.ru ; shavgulidze@bk.ru

arXiv: 1303.3523(math-ph);
arXiv: 1303.5027(hep-th)

Let us consider the model
of ϕ^4 self-interacting quantum field
in one-dimensional "space-time".
Under the non-linear non-local substitution

$$\chi(t) = \phi(t) + \int\limits_0^t \phi^2(\tau)d\tau \tag{1}$$

it turns into the free field model.
And we get
the formal equality of the functional integrals

$$\int \exp\left\{-\frac{1}{2}\int\limits_0^1 (\dot{\chi}(t))^2 dt\right\}\ d\chi\ =$$

$$\int \exp\left\{-\frac{1}{2}\int\limits_0^1 (\dot{\phi}(t))^2 dt - \frac{1}{2}\int\limits_0^1 \phi^4(t)dt - \int\limits_{t=0}^{t=1} \phi^2(t)d\phi\right\}\ d\phi\,. \tag{2}$$

In eq. (2), the last term in the exponent
is the well-known Ito stochastic integral

$$\int\limits_{t=0}^{t=1} \phi^2(t)d\phi = \tag{3}$$

$$\frac{1}{3}\left[\phi^3(1) - \phi^3(0)\right] - \int\limits_0^1 \phi(t)dt\,.$$

Note that functions $\chi(t)$ and $\phi(t)$
belong to different functional spaces.
Namely, if

$$\chi(t) \in C([0,1])\,,$$

than

$$\phi(t) \in X \neq C([0,1])\,.$$

Here, X is the space of functions
that can have a finite number
of singularities of the type

$$(t - t_i^*)^{-1}$$

on the interval $[0, 1]$.
It is quite natural
as for a bounded function $\chi(t)$
there can be some regions
where $|\varphi(t)|$ becomes large enough.
And in these regions
the behavior of the function $\phi(t)$
is prescribed by the equation $\dot{\phi} = -\phi^2$.

Thus, the equivalence of the two theories
manifests itself in the
equality of the functional integrals

$$Z = \int\limits_{C[0,1]} \exp\left\{-\frac{1}{2}\int\limits_0^1 (\dot{\chi}(t))^2 dt\right\}\, d\chi =$$

$$\int\limits_X \exp\left\{-\frac{1}{2}\int\limits_0^1 (\dot{\phi}(t))^2 dt - \frac{1}{2}\int\limits_0^1 \phi^4(t) dt\right\} \tag{4}$$

$$\exp\left\{-\frac{1}{3}\left[\phi^3(1) - \phi^3(0)\right] + \int\limits_0^1 \phi(t) dt\right\}\, d\phi\,.$$

We see that for interacting quantum fields
the functional space appears to be more singular
than the one for the free field.
Note that on the space X ,

$$\exp\left\{-\frac{1}{2}\int\limits_0^1 (\dot{\phi}(t))^2 dt\right\}\, d\phi$$

is not a measure.
An attempt to expand the exp
in powers of the interaction ϕ^4
leads to ambiguous infinite expressions.
And only

$$\exp\left\{-\frac{1}{2}\int\limits_0^1 (\dot{\phi}(t))^2 dt - \frac{1}{2}\int\limits_0^1 \phi^4(t) dt\right\}$$

$$\exp\left\{-\frac{1}{3}\left[\phi^3(1) - \phi^3(0)\right] + \int\limits_0^1 \phi(t) dt\right\}\, d\phi$$

can be considered as a measure on X .
The similar picture takes place

for four-dimensional space-time
where the interacting quantum field
is not a continuous function but a distribution.
The Gaussian measure of the set of continuous functions
is equal to zero

$$\int\limits_{C} \exp\left\{-\frac{1}{2}\int\limits_{\mathbf{R}^D} \phi(x)\Delta_{(D)}\phi(x)d^Dx\right\} d\phi = 0\,, \quad D = 2, \ldots\,,$$

and the well-known Haag theorem is valid.
The non-linear non-local substitution (1)
and the equality of the functional integrals (4)
lead to an interesting effect.
The classical action
and the classical limit of the corresponding quantum theory
turn out to be different.
First, let us describe the model.
We slightly modify the interaction term
and consider the action

$$A = \frac{1}{2}\int\limits_{-T}^{+T} (\dot{\varphi}(t))^2 dt + \tag{5}$$

$$\frac{a^2}{2}\int\limits_{-T}^{+T}\left(\varphi^2(t) - \frac{1}{4}\frac{b^2}{a^2}\right)^2 dt\,.$$

The potential

$$V(\varphi(t)) = \frac{1}{2}a^2\left(\varphi^2(t) - \beta^2\right)^2 \tag{6}$$

$$\left(\beta \equiv \frac{b}{2a}\,;\ a > 0,\ b > 0\right)$$

has two degenerate minimae at

$$\varphi = \pm\beta$$

and a local (unstable) maximum at

$$\varphi = 0\,,$$

and is symmetric:

$$V(-\varphi) = V(\varphi)\,.$$

The Euler-Lagrange equation has the form

$$\ddot{\varphi}(t)) - 2a^2\varphi\left(\varphi^2(t) - \beta^2\right) = 0\,. \tag{7}$$

So, the classical system given by the action A
is symmetrical under the substitution

$$\varphi \to -\varphi\,.$$

For the classical system given by the action

$$A_+ = A - a\int\limits_{-T}^{+T}\varphi(t)dt + \tag{8}$$

$$\frac{a}{3}\left[\varphi^3(+T) - \varphi^3(-T)\right] - a\beta^2\left[\varphi(+T) - \varphi(-T)\right]$$

the symmetry is broken because of the term
linear in φ and the boundary terms.
The action A_+ leads to the Euler-Lagrange equation

$$\ddot{\varphi}(t)) - 2a^2\varphi\left(\varphi^2(t) - \beta^2\right) + a = 0 \tag{9}$$

and the boundary conditions

$$\dot{\varphi}(\pm T) + a\left(\varphi^2(\pm T) - \beta^2\right) = 0\,. \tag{10}$$

The corresponding quantum theory
deals with the functional measure

$$\int \exp\{-A_+(\varphi)\}d\varphi\,.$$

Now the substitution

$$\chi(t) = \varphi(t) + \tag{11}$$

$$a \int\limits_{-T}^{t} \left(\varphi^2(\tau) - \beta^2\right) d\tau$$

results in the equality of the functional integrals

$$\int\limits_{X^+} F(\varphi) \exp\{-A_+(\varphi)\} d\varphi = \tag{12}$$

$$\int\limits_{C[-T,+T]} F(\varphi(\chi)) \exp\left\{-\frac{1}{2} \int\limits_{-T}^{+T} (\dot{\chi}(t))^2 dt\right\} d\chi \,.$$

The functional space X^+
is the space of functions
that can have singularities
on the interval $[-T, +T]$.
Note that the term linear in φ
and breaking the symmetry
appears in the integrand
of the left-hand side of eq. (12)
from the Ito integral (see eq. (3)).
Now consider the classical limit of eq. (12)

$$F(\tilde{\varphi}) =$$

$$\lim_{\hbar \to 0} \int\limits_{X^+} F(\varphi) \exp\{-\frac{1}{\hbar} A_+(\varphi)\} d\varphi =$$

$$\lim_{\hbar \to 0} \int\limits_{C[-T,+T]} F(\varphi(\chi)) \tag{13}$$

$$\exp\left\{-\frac{1}{\hbar} \frac{1}{2} \int\limits_{-T}^{+T} (\dot{\chi}(t))^2 dt\right\} d\chi \,.$$

In the right hand side of eq. (13)
the classical limit yields the equation

$$\ddot{\chi}(t) = 0\,,$$

or

$$\dot{\chi}(t) = const\,.$$

Actually, $const = 0$ as $\chi(+T)$ is not fixed.
It follows from the boundary conditions (10) as well.
In terms of the function $\varphi(t)$
the equation looks like

$$\dot{\varphi}(t) + a\left(\varphi^2(t) - \beta^2\right) = 0\,. \tag{14}$$

Note that eq. (7) can be represented in the form

$$\dot{\varphi} = \pm\sqrt{f(\varphi)}$$

and has two branches of solutions
corresponding to the different signs.
It can be easily seen that eq. (14)
is the certain branch of eq. (7)
with the proper integration constant.
Thus, a solution of eq. (14)

$$\tilde{\varphi}^+(t)$$

is a solution of eq. (7) but not one of eq. (9)!
In this sense,
quantum theory restores the symmetry
broken in classical theory.
The crucial point here
is the integration over the functional space X^+

containing singular functions.
If it were the space C
we would get the ordinary result
obtained after integration
over the Wiener measure

$$\exp\left\{-\frac{1}{2}\int_0^1 (\dot{\varphi}(t))^2 dt\right\}\, d\varphi\,.$$

The explicit form of the solution $\tilde{\varphi}^+(t)$
can be easily found.
Depending on the bound value

$$\tilde{\varphi}^+(-T) \equiv -\alpha$$

it is

$$\tilde{\varphi}^+_\alpha(t) = \beta \tanh(bt + c)\,, \quad for\ -\alpha > -\beta\,, \tag{15}$$

or

$$\tilde{\varphi}^+_\alpha(t) = \beta \coth(bt + c)\,, \quad for\ -\alpha < -\beta\,. \tag{16}$$

If we set the integration constant c
to be equal to zero,
the solution is the odd function

$$\tilde{\varphi}^+(-t) = -\tilde{\varphi}^+(t)\,.$$

In this case,

$$A_+\left(\tilde{\varphi}^+\right) = A\left(\tilde{\varphi}^+\right)\,.$$

So far, we have considered the theory given by the action A_+.
However, the similar picture holds for the "mirror" action

$$A_- = A + a\int_{-T}^{+T} \varphi(t)dt \tag{17}$$

$$-\frac{a}{3}\left[\varphi^3(+T)-\varphi^3(-T)\right]+a\beta^2\left[\varphi(+T)-\varphi(-T)\right] .$$

The corresponding substitution is

$$\chi(t)=\varphi(t) \tag{18}$$

$$-a\int\limits_{-T}^{t}\left(\varphi^2(\tau)-\beta^2\right)d\tau .$$

And we have the equality of the functional integrals

$$\int\limits_{X^-} F(\varphi)\, \exp\{-A_-(\varphi)\}d\varphi = \tag{19}$$

$$\int\limits_{C[-T,+T]} F(\varphi(\chi))\, \exp\left\{-\frac{1}{2}\int\limits_{-T}^{+T}(\dot\chi(t))^2dt\right\}\, d\chi .$$

The structure of the space X^-
is the same as of the space X^+.
But the functions from X^+ and the functions from X^-
have singularities at different points.
The solutions $\tilde\varphi^-(t)$ of the equation of motion

$$\dot\varphi(t)+a\left(\varphi^2(t)-\beta^2\right)=0 \tag{20}$$

obtained in the classical limit
belong to the other branch of the solutions of eq. (7).
They are connected with the solutions (15), (16)
by the substitution $t\to -t$.
For an invertible substitution

$$\xi(t)=\varphi(t)+\int\limits_0^t f(\varphi(\tau))d\tau \tag{21}$$

the equality

$$\int F(\xi)\exp\left\{-\frac{1}{2}\int\limits_0^1(\dot\xi(t))^2dt\right\}\, d\xi\ =$$

$$\int F(\xi(\varphi))$$

$$\exp\left\{-\frac{1}{2}\int\limits_0^T \Big((\dot\varphi(t))^2 + f^2(\varphi(t)) - f'(\varphi(t))\Big)\,dt \,+\, b.t.\right\}\, d\varphi \quad (22)$$

is valid.

For the model of quantum scalar field with the classical action

$$\tilde{A}(\varphi) =$$

$$\frac{1}{2}\int\limits_0^T \Big\{(\dot\varphi(t))^2 dt - \alpha\lambda\, e^{\alpha\varphi(t)} + \lambda^2\, e^{2\alpha\varphi(t)}\Big\}\, dt \quad (23)$$

we can get the explicit form of the function $\varphi(\xi)$

$$\varphi(t) = \xi(t) - \frac{1}{\alpha}\ln\left(\alpha\lambda\int\limits_0^t e^{\alpha\xi(\tau)}d\tau\right)\,. \quad (24)$$

The scale factor

$$a(t) = e^{\varphi(t)}$$

in terms of $\xi(t)$ looks like

$$a(t) = e^{\xi(t)}\left(\alpha\lambda\int\limits_0^t e^{\alpha\xi(\tau)}d\tau\right)^{-\frac{1}{\alpha}}\,. \quad (25)$$

Now, in the quantum theory given by the action $\tilde{A}(\varphi)$ we can evaluate the mean value of the scale factor, moments and other quantities connected with it.