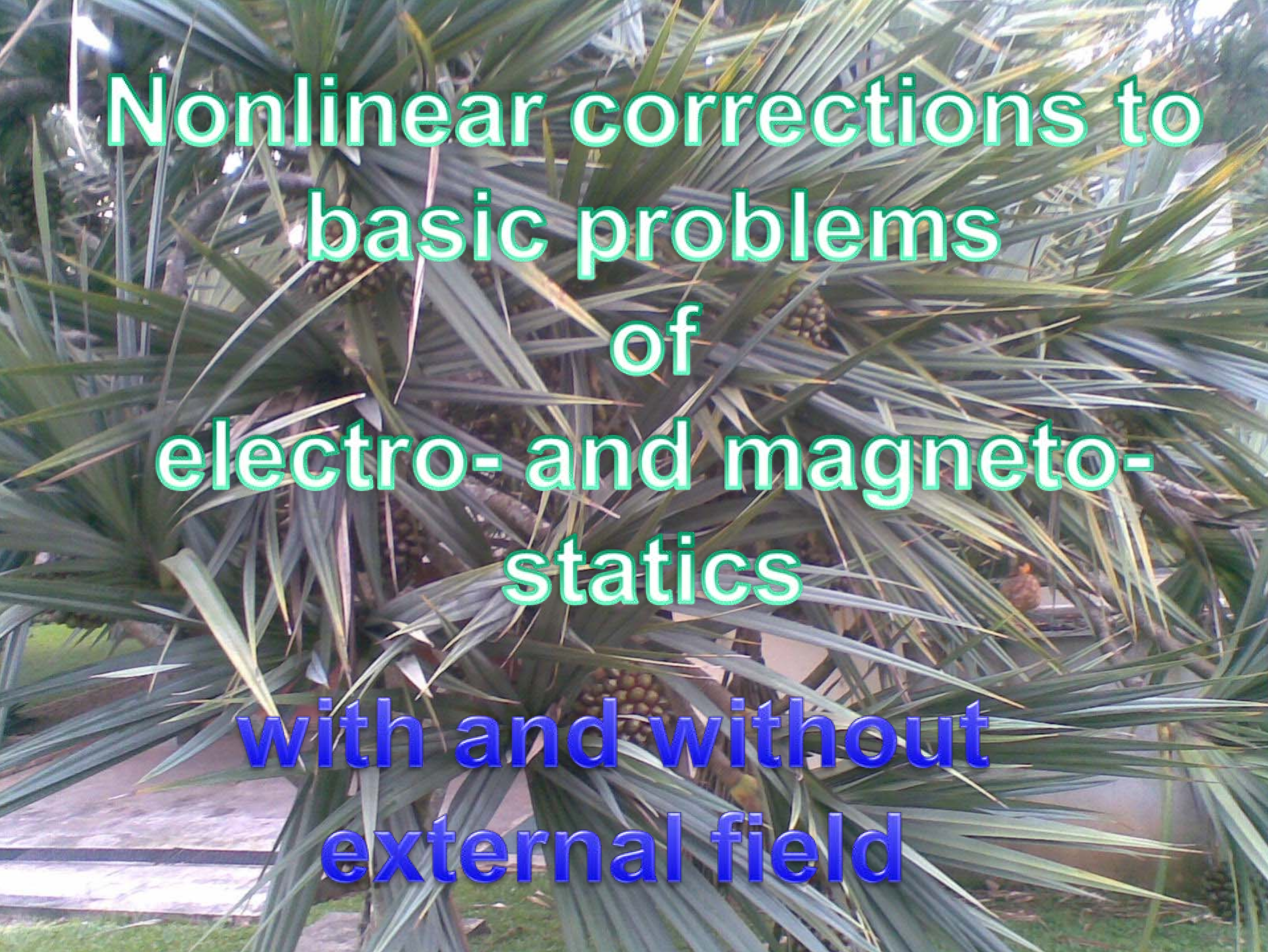

Nonlinear corrections to basic problems of electro- and magneto-statics

with and without external field

Nonlinear corrections to basic problems of electro- and magneto-statics with and without external field

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Nonlinear Maxwell equations

truncated at the fourth power of the field $a^{\tau}(x)$

$$j_{\mu}(x) = [\Box\eta_{\mu\tau} - \partial_{\mu}\partial_{\tau}]\, a^{\tau}(x) + \int \mathrm{d}^4y\, \Pi_{\mu\tau}(x,y)\, a^{\tau}(y)$$
$$+\frac{1}{2}\int \mathrm{d}^4y\mathrm{d}^4u\, \Pi_{\mu\tau\sigma}(x,y,u)\, a^{\tau}(y)\, a^{\sigma}(u)$$
$$+\frac{1}{6}\int \mathrm{d}^4y\mathrm{d}^4u\mathrm{d}^4v\, \Pi_{\mu\tau\sigma\rho}(x,y,u,v)\, a^{\tau}(y)\, a^{\sigma}(u)\, a^{\rho}(v)$$

$$A_{\beta}(x) = \mathcal{A}_{\beta}^{\mathrm{ext}}(x) + a_{\beta}(x)$$

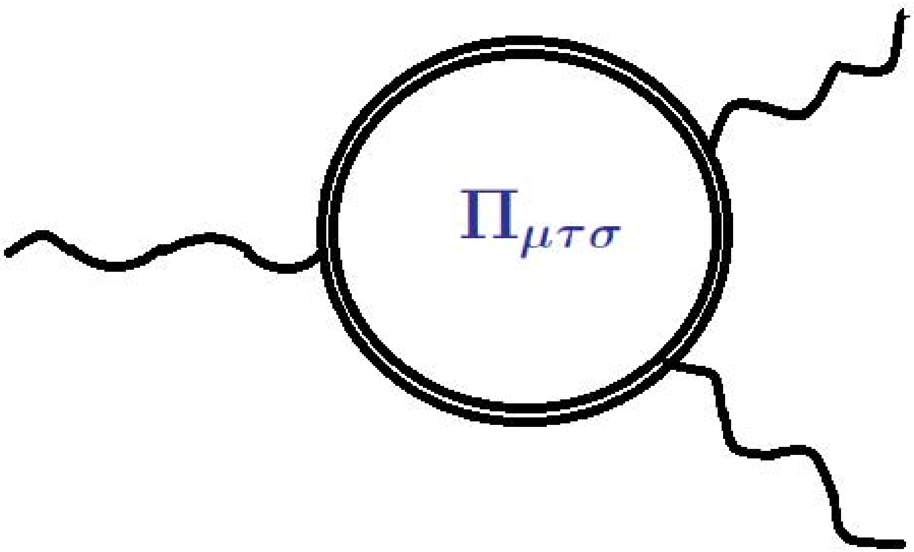
Higher-rank polarization tensors

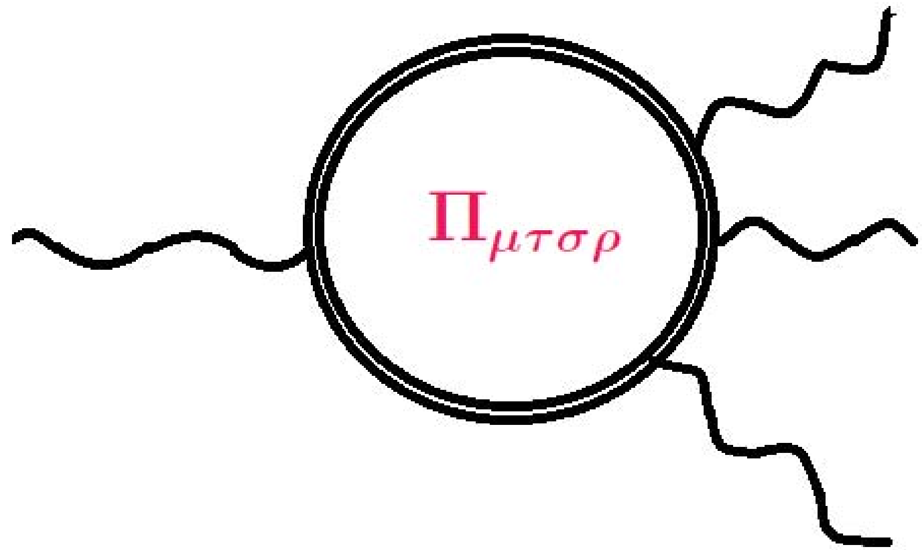
$$\Pi_{\mu\tau}\left(x,x'\right)=\left.\frac{\delta^2\Gamma}{\delta A^\mu\left(x\right)\delta A^\tau\left(x'\right)}\right|_{A=\mathcal{A}^{\text{ext}}},$$

$$\Pi_{\mu\tau\sigma}\left(x,x',x''\right)=\left.\frac{\delta^3\Gamma}{\delta A^\mu\left(x\right)\delta A^\tau\left(x'\right)\delta A^\sigma\left(x''\right)}\right|_{A=\mathcal{A}^{\text{ext}}},$$

$$\Pi_{\mu\tau\sigma\rho}\left(x,x',x'',x'''\right)=\left.\frac{\delta^4\Gamma}{\delta A^\mu\left(x\right)\delta A^\tau\left(x'\right)\delta A^\sigma\left(x''\right)\delta A^\rho\left(x'''\right)}\right|_{A=\mathcal{A}^{\text{ext}}}$$

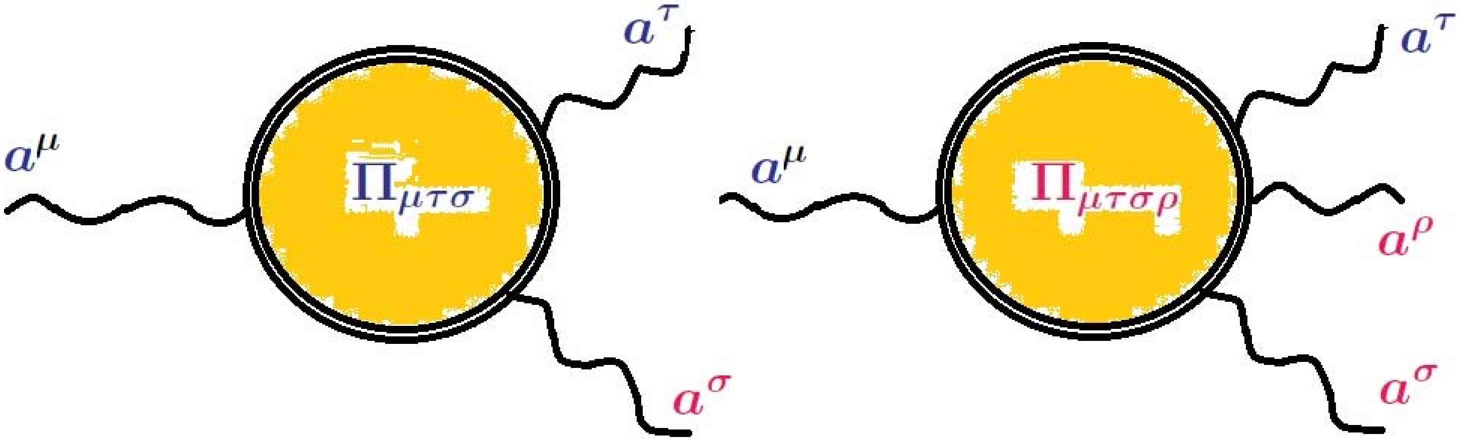
$\Gamma[A]=\int\mathcal{L}(x)\mathrm{d}^4x$ is the effective action functional


$\Pi_{\mu\tau\sigma}$


$\Pi_{\mu\tau\sigma\rho}$

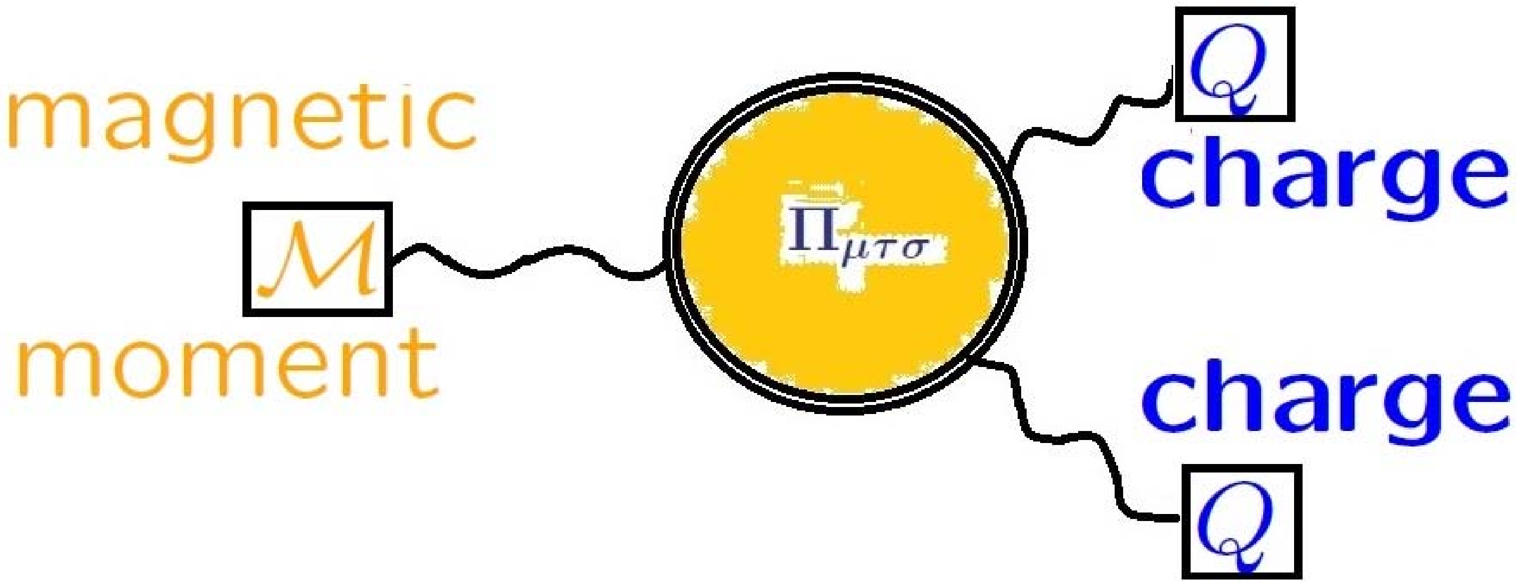
Self-interaction +

interaction with external field

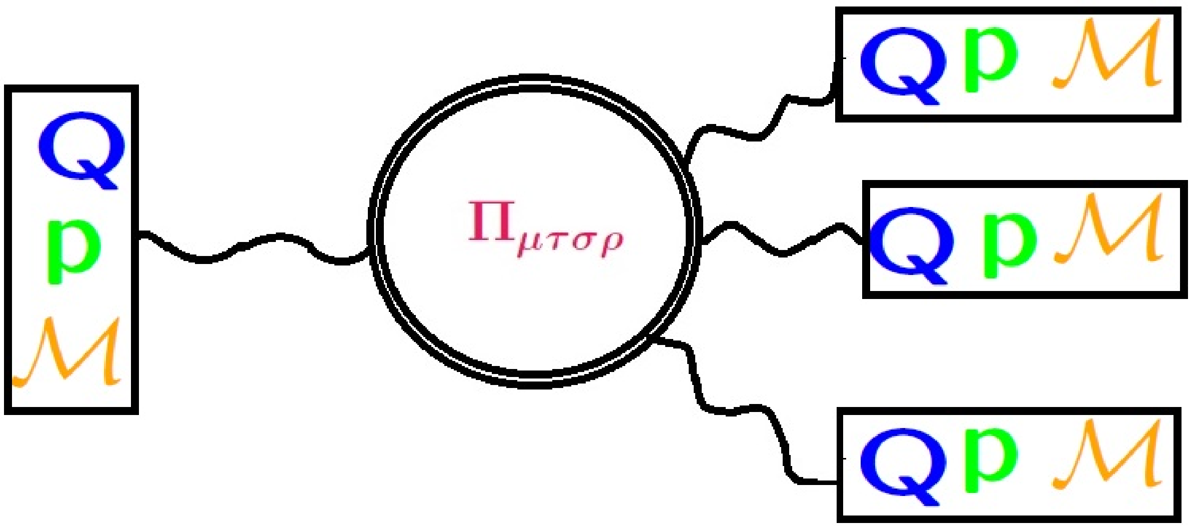


I need them off-shell

Magnetic (dipole) field produced by static charge



Cubic corrections to the fields of charge $\mathbf{Q}$, electric $\mathbf{p}$ and magnetic $\mathcal{M}$ dipole



Infrared approximation

$$\Gamma\left[A\right] = \int \mathcal{L}\left(z\right) d^4 z = \Gamma\left[\mathfrak{F}, \mathfrak{G}, \cancel{\partial_x F_{\mu\nu}, \ldots}\right]$$

$$\mathfrak{F}\left(z\right) = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}\,, \;\; \mathfrak{G} = \frac{1}{4} F^{\rho\sigma} \tilde{F}_{\rho\sigma}$$

$$F^{\mu\nu} = \partial_\mu A^\nu(x) - \partial_\nu A^\mu(x),$$

$$\tilde{F}_{\rho\sigma} = \frac{1}{2} \epsilon_{\rho\sigma\lambda\kappa} F^{\lambda\kappa}$$

$$\frac{\delta^4\Gamma}{A^{\mu}(x)\,\delta A^{\tau}(y)\,\delta A^{\sigma}(u)\,\delta A^{\rho}(v)} =$$

$$= \int \mathrm{d}^4 z \,\left\{\frac{\partial^2\mathcal{L}}{\partial\mathfrak{F}^2}[(\eta_{\alpha\lambda}\eta_{\rho\mu}-\eta_{\mu\lambda}\eta_{\alpha\rho})\,(\eta_{\tau\sigma}\eta_{\beta\gamma}-\eta_{\beta\sigma}\eta_{\tau\gamma})+...] + \frac{\partial^2\mathcal{L}}{\partial\mathfrak{G}^2}[\epsilon_{\alpha\mu\beta\tau}\epsilon_{\lambda\rho\gamma\sigma}\right.$$

$$+ \frac{\partial^3\mathcal{L}}{\partial\mathfrak{F}^3}[F_{\alpha\mu}F_{\beta\tau}\,(\eta_{\gamma\lambda}\eta_{\rho\sigma}-\eta_{\sigma\lambda}\eta_{\gamma\rho})+...] + \frac{\partial^3\mathcal{L}}{\partial\mathfrak{F}\partial\mathfrak{G}^2}\,[\tilde{F}_{\alpha\mu}\tilde{F}_{\beta\tau}\,(\eta_{\gamma\lambda}\eta_{\rho\sigma}-\eta_{\sigma\lambda}\eta_{\gamma\rho})$$

$$+ \frac{\partial^4\mathcal{L}}{\partial\mathfrak{F}^2\partial\mathfrak{G}^2}\,[F_{\gamma\sigma}F_{\lambda\rho}\tilde{F}_{\alpha\mu}\tilde{F}_{\beta\tau}+ ...] + \frac{\partial^4\mathcal{L}}{\partial\mathfrak{F}^4}[F_{\alpha\mu}F_{\beta\tau}F_{\gamma\sigma}F_{\lambda\rho}]$$

$$\left.+ \frac{\partial^4\mathcal{L}}{\partial\mathfrak{G}^4}\tilde{F}_{\alpha\mu}\tilde{F}_{\beta\tau}\tilde{F}_{\gamma\sigma}\tilde{F}_{\lambda\rho}\right\}\frac{\partial\delta^4(x-z)}{\partial z_\alpha}\frac{\partial\delta^4(y-z)}{\partial z_\beta}\frac{\partial\delta^4(u-z)}{\partial z_\gamma}\frac{\partial\delta^4(v-z)}{\partial z_\lambda}.$$

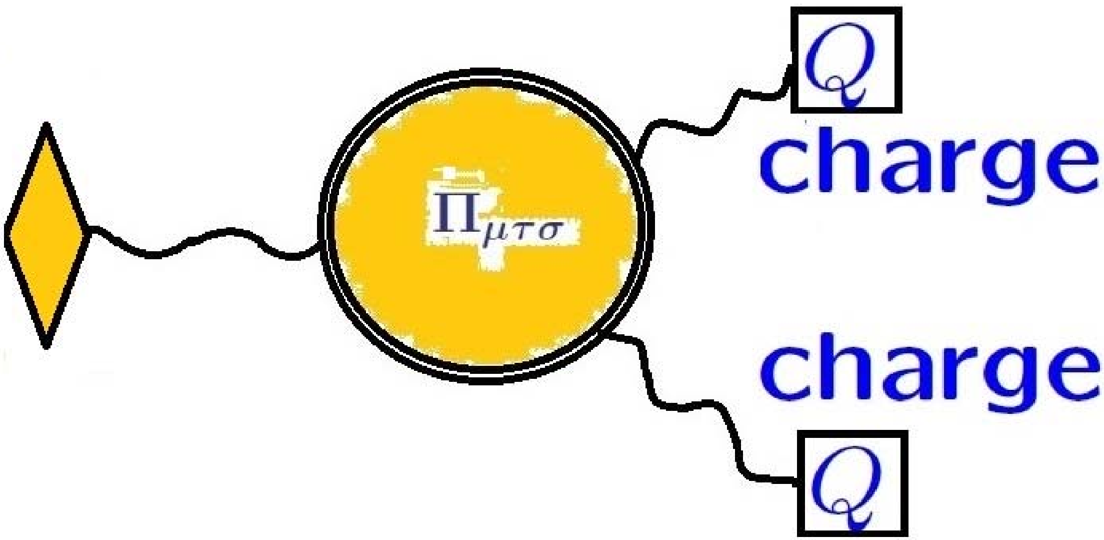
Linear correction to Coulomb field in Magnetic field

$$\mathfrak{F} > 0, \mathfrak{G} = 0$$

Far from the charge Q

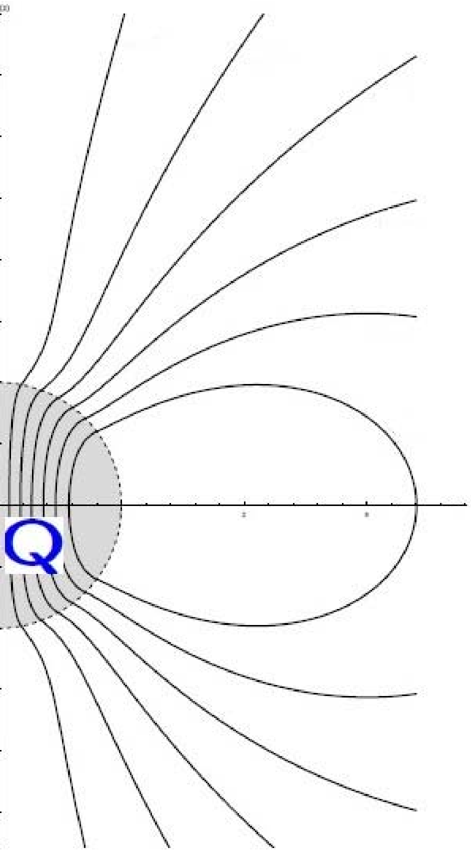
$$1 - \mathcal{L}_{\mathfrak{F}} = \varepsilon_{\text{tr}}, \quad 1 - \mathcal{L}_{\mathfrak{F}} + 2\mathfrak{F}\mathcal{L}_{\mathfrak{G}\mathfrak{G}} = \varepsilon_{\text{long}},$$

$$a^0_{\text{lin}}(\mathrm{x}) = \frac{1}{4\pi\varepsilon_{\text{tr}}^{1/2}} \frac{Q}{\left(\mathrm{x}_\perp{}^2\,\varepsilon_{\text{long}} + x_3{}^2\,\varepsilon_{\text{tr}}\right)^{1/2}}$$



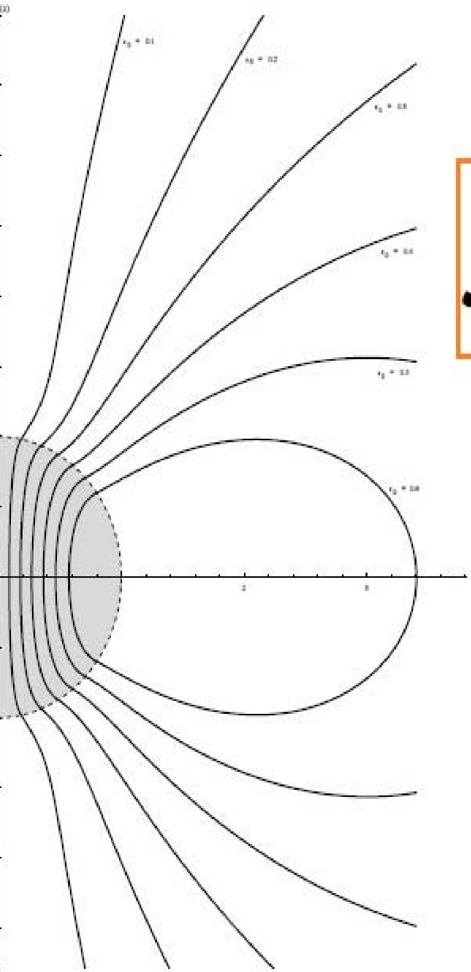
Quadratic correction to the Coulomb field in a magnetic field

is purely magnetic



Quadratic correction to the Coulomb field in a magnetic field is purely magnetic

Charge
Q
homogeneously distributed over a sphere



Magnetic moment

$$\mathcal{M} = \frac{Q^2}{5a}\left(3\mathcal{L}_{\mathfrak{F}\mathfrak{F}} - 2\mathcal{L}_{\mathfrak{G}\mathfrak{G}} - B^2\mathcal{L}_{\mathfrak{F}\mathfrak{G}\mathfrak{G}}\right)\boldsymbol{B}$$

of charge Q

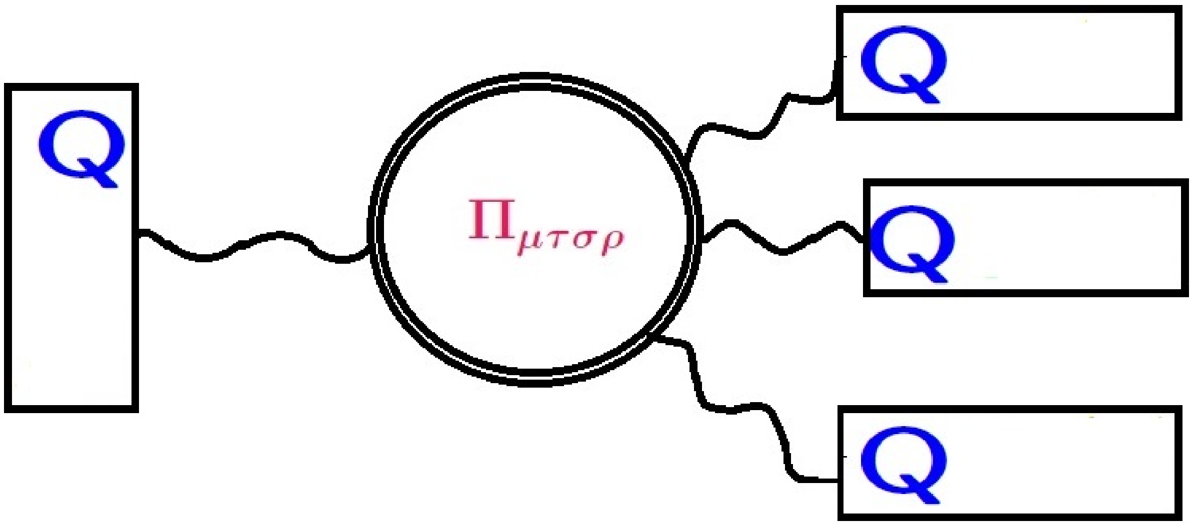
distributed over sphere of radius a

in magnetic field $\boldsymbol{B}$

in terms of derivatives of effective Lagrangian over invariants

$$\mathfrak{F} = \frac{B^2}{2},\ \mathfrak{G} = 0$$

Cubic corrections to the field of charge



$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{lin}}(\mathbf{r}) - \frac{1}{2}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\mathbf{E}(\mathbf{r})\, E^2(r)$$

cubic equation for spheric–symmetric electric field $\boldsymbol{E} = \frac{\boldsymbol{r}}{r}E(r)$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{lin}}(\mathbf{r}) - \frac{1}{2}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\mathbf{E}(\mathbf{r})\,E^2(r)\,.$$

Iteration: $\mathbf{E}(\mathbf{r}) \simeq \mathbf{E}^{\mathrm{lin}}(\mathbf{r})$ in r-h side

With the Euler-Heisenberg Lagrangian $\mathcal{L}_{\mathfrak{F}\mathfrak{F}} = \frac{e^4}{45\pi^2 m^4}$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{lin}}(\mathbf{r})\left(1 - \frac{2\alpha}{45\pi}\left(\frac{eE^{\mathrm{lin}}(r)}{m^2}\right)^2\right)$$

Point charge $E^{\mathrm{lin}}(r) = \frac{Q}{4\pi r^2}$

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi r^2}\left(1 - \frac{2\alpha}{45\pi}\left(\frac{e\,Q}{4\pi r^2\, m^2}\right)^2\right)$$

Wichmann and Kroll 1954

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{lin}}(\mathbf{r}) - \frac{1}{2}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\mathbf{E}(\mathbf{r})\,E^2(r)$$

cubic equation for spheric–symmetric electric field $\boldsymbol{E} = \dfrac{\boldsymbol{r}}{r}E(r)$

$$\cancel{E(r)} = E^{\text{lin}}(r) - \frac{1}{2}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}(r)\,E^3(r)$$

$$E(r) \ll E^{\text{lin}}(r)$$

$$E(r) \simeq \left(\frac{2E^{\text{lin}}(r)}{\mathcal{L}_{\mathfrak{F}\mathfrak{F}}}\right)^{\frac{1}{3}} \simeq \left(\frac{2\,Q}{4\pi\, r^2\mathcal{L}_{\mathfrak{F}\mathfrak{F}}}\right)^{\frac{1}{3}} \sim r^{-\frac{2}{3}}$$

convergence of field energy of charge Q

Cubic corrections to magnetic dipole field

current, producing dipole field $\boldsymbol{j}(\boldsymbol{r}) = 3\frac{\boldsymbol{\mathcal{M}}^{\mathrm{lin}} \times \mathbf{r}}{r^4}\delta(r-a)$

Dipole field reproduces itself in the remote region

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}^{\mathrm{lin}} - \frac{7}{5}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\boldsymbol{\mathcal{M}}\left(\frac{\mathcal{M}}{a^3}\right)^2$$

Cubic corrections to electric dipole field

current, producing dipole field $j_0(\mathbf{r}) = 3\frac{\mathbf{r}\cdot\mathbf{p}^{\mathrm{lin}}}{r^4}\delta(r-a)$

Dipole field reproduces itself in the remote region

$$\mathbf{p} = \mathbf{p}^{\mathrm{lin}} - \frac{\mathbf{p}}{10}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\left(\frac{p}{a^3}\right)^2$$


Thank you for attention