Higher-Loop Calculations of the UV to IR Evolution of Gauge Theories and Remarks on Neutrino Properties

Robert Shrock

YITP, Stony Brook University

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Outline

- Neutrino masses and loop calculations of electromagnetic properties
- Renormalization-group flow in an asymptotically free gauge theory from UV to IR; types of IR behavior; role of an exact or approximate IR fixed point
- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension γ_m of fermion bilinear
- Some comparisons with lattice measurements of γ_m
- Higher-loop calculation of structural properties of β
- Results in the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed
- Study of scheme-dependence
- Conclusions

Neutrino masses and loop calculations of electromagnetic properties

Neutrino masses and lepton mixing are confirmed evidence of physics beyond the Standard Model (SM), pioneering work by Pontecorvo; Gribov and Pontecorvo; Maki, Nakagawa, Sakata on searches for these.

Neutrino oscillations discovered in solar and atmospheric ν experiments (Davis Homestake exp., Kamiokande, IMB, SAGE, GALLEX, SuperKamiokande, SNO) further studies via accelerator exps. (K2K, MINOS, MiniBooNE, OPERA, T2K, NOvA...) and reactor exps. (KamLAND, Double Chooz, Daya Bay, RENO...); great progress in gaining knowledge of ν masses and lepton mixing, with intensive current and future exp. programs; many talks at this conf.

One extends the SM to include neutrino masses via addition of electroweak-singlet $\nu_{i,R}$ fields, and hence Dirac and Majorana mass terms with interesting connection to possible UV completions of SM.

Via seesaw mechanism, very small ν masses are plausibly related to very high mass scales of new physics beyond SM.

Questions still to be answered include mass hierarchy, leptonic CP violation, possibility of light, primarily electroweak-singlet (sterile) neutrinos (LSND,..)

In addition to ν oscillations, ν masses and lepton mixing lead to decays such as $\mu \to e\gamma$, $\mu \to ee\bar{e}$ (in m_ν-extended SM, Bilenky, Petcov, and Pontecorvo, Marciano and Sanda, Lee, Pakvasa, Shrock, Sugawara (1977)) However, leptonic GIM mechanism produces extremely small branching ratio.

Even with zero electric charge, neutrinos have induced diagonal and off-diagonal interactions with photons via one-loop diagrams.

These can mediate one-loop induced decays (e.g., Shrock (1974); 1977 papers...)

Dirac neutrino has a magnetic dipole moment (Fujikawa and Shrock, 1980; from general one-loop formulas)

$$\mu_\nu = \frac{3eG_F m_\nu}{8\pi^2\sqrt{2}} = (3.2 \times 10^{-19})\Big(\frac{m_\nu}{1\ \mathrm{eV}}\Big)\mu_B$$

where $\mu_B = e/(2m_e)$. ν can also have CPV electric dipole moment. Majorana ν has transition dipole moments (diagonal ones vanish).

Detailed studies of effects on neutrino scattering - Studenikin, Voloshin,...

Limits from reactor $\bar{\nu}_e$ exps (Savannah River, Krasnoyarsk, Rovno, MUNU, TEXONO, GEMMA..): $\mu_\nu < (3 \times 10^{-11})\mu_B$; also limits from LSND, SuperK, Borexino... Astrophysical limit from cooling of red giants: $\mu_\nu < (3 \times 10^{-12})\mu_B$.

Recent review of neutrino EM ν properties: Broggini, Giunti, Studenikin, arXiv:1207.3980.

Searches for ν masses from nuclear beta decay exps (LANL, Tokyo, Zurich, Mainz, Moscow-Troitsk) continue to reduce limits on $[\sum_i |U_{ei}|^2 m(\nu_i)^2]^{1/2}$ to $\lesssim 2$ eV (future, KATRIN...). Stringent cosmological upper limit on $\sum_i m(\nu_i)$.

Also useful to search for emission of heavier neutrinos via mixing (Shrock, Phys. Lett. B96, 159 (1980); Kobzarev, Martemyanov, Okun, Shchepkin, Yad. Fiz. 32, 1590 (1980) [Sov. J. Nucl. Phys. 32, 823 (1980)] via kink in Kurie plot, correlated upper limits on $|U_{ei}|^2$ from many searches; e.g., recent Troitsk limits in Belesev et al., JETP Letts. 97, 67 (2013) [arXiv:1211.7193], Belesev et al., arXiv:1307.5687.

Neutrino masses, mixing, and electromagnetic properties of neutrinos continue to be important areas of theoretical and experimental study, especially since they probe new physics beyond the SM.

The rest of this talk includes material from the following papers, and some new results:

- Ryttov and Shrock, “Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions”, Phys. Rev. D 83, 056011 (2011), arXiv:1011.4542

- Ryttov and Shrock, “Scheme Transformations in the Vicinity of an Infrared Fixed Point”, Phys. Rev. D 86, 065032 (2012), arXiv:1206.2366; “An Analysis of Scheme Transformations in the Vicinity of an Infrared Fixed Point”, Phys. Rev. D 86, 085005 (2012), arXiv:1206.6895

- Shrock, “Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories”, Phys. Rev. D 87, 105005 (2013), arXiv:1301.3209

- Shrock, “Higher-Loop Calculations of the Ultraviolet to Infrared Evolution of a Vectorial Gauge Theory in the Limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c Fixed”, Phys. Rev. D 87, 116007 (2013), arXiv:1302.5434

- Shrock, ‘Study of Scheme Transformations to Remove Higher-Loop Terms in the β Function of a Gauge Theory”, Phys. Rev. D 88, 036003 (2013), arXiv:1305.6524.

RG Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV). The quark-parton picture is applicable here and leads to scaling behavior in deep inelastic scattering, and quark-counting rules for $d\sigma/dt$ and $F(t)$ (Matveev, Muradyan, and Takhelidze; Brodsky and Farrar).

The question of how an asymptotically free gauge theory flows from large μ in the UV to small μ in the infrared (IR) depends on N_f and is of fundamental field-theoretic interest.

In QCD with $N_f = 2$ or $N_f = 3$ light quarks, the β function has no perturbative IR zero. We focus on a theory with larger N_f and hence different properties, where there may be an exact or approximate IR fixed point (zero of β).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

The dependence of $\alpha(\mu)$ on μ is described by the renormalization group (Stueckelberg and Peterman, Gell-Mann and Low, Bogoliubov, Shirkov, Callan, Symanzik, Wilson). The β function is

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ ,$$

where $t = \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \ge 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region. This is the case for QCD.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \longrightarrow \alpha_{IR}$. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG); as $\mu \longrightarrow 0$ and $\alpha \longrightarrow \alpha_{IR}$, $\beta \longrightarrow \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions.

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant (= dilatation-invariant) behavior.

SχSB at Λ also breaks the approx. dilatation symmetry, might lead to a resultant approx. NGB, the dilaton. This is not massless, since $\beta(\alpha_{cr})$ is nonzero.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$.

At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$. One-loop coefficient b_1 is (Gross and Wilczek, Politzer)

$$b_1 = \frac{1}{3}(11C_A - 4N_f T_f)$$

where $C_A \equiv C_2(G)$ is quadratic Casimir invariant, $T_f \equiv T(R)$ is trace invariant. Focus here on $G = \mathrm{SU}(N_c)$.

Asymp. freedom requires $N_f < N_{f,b1z}$, where

$$N_{f,b1z} = \frac{11C_A}{4T_f}$$

e.g., for $R =$ fundamental rep., $N_f < (11/2)N_c$.

Two-loop coeff. b_2 is (with $C_f \equiv C_2(R)$) (Caswell, Jones)

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)N_f\, T_f\right]$$

For small N_f, $b_2 > 0$; b_2 decreases as fn. of N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)}$$

For arbitrary G and R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval in N_f for which β has an IR zero, namely

$$I : \quad N_{f,b2z} < N_f < N_{f,b1z}$$

- for SU(2), I: $\quad 5.55 < N_f < 11$
- for SU(3), I: $\quad 8.05 < N_f < 16.5$
- As $N_c \to \infty$, I: $\quad 2.62 N_c < N_f < 5.5 N_c$.

(expressions evaluated for $N_f \in \mathbb{R}$, but it is understood that physical values of N_f are nonnegative integers.)

As N_f decreases from the upper to lower end of interval I, α_{IR} increases. Denote

$$N_f = N_{f,cr} \quad \text{at} \quad \alpha_{IR} = \alpha_{cr}$$

Value of $N_{f,cr}$ is of fundamental importance, since it separates the (zero-temp.) chirally symmetric and broken IR phases.

Intensive current lattice studies of SU(N_c) gauge theories with N_f copies of fermions in various representations R; progress toward determining $N_{f,cr}$ for various N_c and R.

Higher-Loop Corrections to UV $\rightarrow$ IR Evolution of Gauge Theories

Because of this strong-coupling physics, one should calculate the IR zero in β, α_{IR}, and resultant value of γ evaluated at α_{IR} to higher-loop order (Ryttov and Shrock, PRD 83, 056011 (2011), arXiv:1011.4542 and Pica and Sannino, PRD 83, 035013 (2011), arXiv:1011.5917; related work by Gardi, Grunberg, Karliner).

Although coeffs. in β at $\ell \geq 3$ loop order are scheme-dependent, results give a measure of accuracy of the 2-loop calc. of the IR zero of β, and similarly with γ evaluated at this IR zero.

We make use of calculation of β and γ up to 4-loops in $\overline{MS}$ scheme by Vermaseren, Larin, and van Ritbergen.

The value of higher-loop calculations has been amply shown in comparison of QCD predictions with experimental data, e.g., in $\overline{MS}$ scheme. Many contributions by authors originally and/or currently at INR, MSU, JINR: Chetyrkin, Gorishny, Kataev, Larin, Surguladze, Tkachov, Tarasov, Vladimirov, Zharkov...

3-loop coefficient in β function (in $\overline{\mathrm{MS}}$ scheme) (Tarasov, Vladimirov, Zharkov; Larin and Vermaseren)

$$b_3 = \frac{2857}{54}C_A^3 + T_f N_f \left[2C_f^2 - \frac{205}{9}C_A C_f - \frac{1415}{27}C_A^2\right] + (T_f N_f)^2\left[\frac{44}{9}C_f + \frac{158}{27}C_A\right]$$

$b_3 < 0$ for $N_f \in I$. Since $\beta_{3\ell} = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2)$, $\beta_{3\ell} = 0$ away from $\alpha = 0$ at two values:

$$\alpha = \frac{2\pi}{b_3}\Big(- b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 < 0$ and $b_3 < 0$, can rewrite as

$$\alpha = \frac{2\pi}{|b_3|}\Big(- |b_2| \mp \sqrt{b_2^2 + 4b_1 |b_3|}\ \Big)$$

Soln. with $-$ sqrt is negative, hence unphysical; soln. with $+$ sqrt is $\alpha_{IR,3\ell}$.

We showed that with this $b_3 < 0$, the value of the IR zero decreases when calculated at the 3-loop level, i.e.,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

Proof:

$$\begin{aligned}\alpha_{IR,2\ell} - \alpha_{IR,3\ell} &= \frac{4\pi b_1}{|b_2|} - \frac{2\pi}{|b_3|}\Big(- |b_2| + \sqrt{b_2^2 + 4b_1|b_3|}\ \Big) \\ &= \frac{2\pi}{|b_2 b_3|}\Big[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\ \Big]\end{aligned}$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0$$

This difference is equal to the positive-definite quantity $4b_1^2b_3^2$, which proves the inequality.

In RS, Phys. Rev. D 87, 105005 (2013), arXiv:1301.3209 we have generalized this.

If a scheme had $b_3 > 0$ in I, then, since $b_2 \to 0$ at lower end of I, $b_2^2 - 4b_1b_3 < 0$ in sqrt, so this scheme would not have a physical $\alpha_{IR,3\ell}$ in this region.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$.

So the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in all such schemes, not just in $\overline{\mathrm{MS}}$.

The 4-loop β function is $\beta = -[\alpha^2/(2\pi)](b_1 + b_2a + b_3a^2 + b_4a^3)$, so $\beta_{4\ell}$ has three zeros away from $\alpha = 0$; smallest (real positive) one as $\alpha_{IR,4\ell}$.

We give an analysis of the zeros of $\beta_{4\ell}$ in a general scheme in Phys. Rev. D 87, 105005 (2013). With $\overline{\mathrm{MS}}$, from 3- to 4-loop level, slight increase: $\alpha_{IR,4\ell} \gtrsim \alpha_{IR,3\ell}$; small change, so overall, $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$.

Our result of smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calc. to higher loop order should give more stable result.

Numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and fermions in fund. rep.

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	6	11.42	1.645	2.395
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(Perturbative calc. not applicable if $\alpha_{IR,n\ell}$ too large.) We have performed the corresponding higher-loop calculations for SU(N_c) gauge theories with N_f fermions in the adjoint, symmetric and antisymmetric rank-2 tensor representations.

We prove a general result on the shift of an IR zero of β when calculated at next higher order: assume fermion content is such that $b_2 < 0$, so theory has a 2-loop IR zero.

Consider a scheme in which the b_ℓ with $\ell = 3, ..., n+1$ have values that preserve the existence of the scheme-independent 2-loop IR zero of β at higher-loop level (motivated physically).

Use fact that theory is asymptotically free, so $\beta < 0$ for $0 < \alpha < \alpha_{IR}$, and hence $d\beta_{n\ell}/d\alpha > 0$ for $\alpha \simeq \alpha_{IR,n\ell}$.

Expand $\beta_{n\ell}$ in Taylor series around $\alpha = \alpha_{IR,n\ell}$:

$$\beta_{n\ell} = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

Now calculate β to the next-higher-loop order, i.e., $\beta_{(n+1)\ell}$, and solve for $\alpha_{IR,(n+1)\ell}$. To determine whether $\alpha_{IR,(n+1)\ell}$ is larger or smaller than $\alpha_{IR,n\ell}$, consider

$$\beta_{(n+1)\ell} - \beta_{n\ell} = -2\bar{b}_{n+1}\alpha^{n+2}$$

In a scheme where $b_{n+1} > 0$, this difference, evaluated at $\alpha = \alpha_{IR,n\ell}$, is negative, so, given that $d\beta_{n\ell}/d\alpha|_{\alpha_{IR,n\ell}} > 0$, to compensate for this, the zero shifts to the right, whereas if $b_{n+1} < 0$, the difference is positive, so the zero shifts to the left.

$$\textbf{If } b_{n+1} > 0\ , \quad \textbf{then } \alpha_{IR,(n+1)\ell} > \alpha_{IR,n\ell}$$

$$\textbf{If } b_{n+1} < 0\ , \quad \textbf{then } \alpha_{IR,(n+1)\ell} < \alpha_{IR,n\ell}$$

This general result is evident in our $\overline{\mathrm{MS}}$ calculations.

$$b_3 < 0, \quad \Longrightarrow \quad \alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

$$b_4 > 0, \quad \Longrightarrow \quad \alpha_{IR,4\ell} > \alpha_{IR,3\ell}$$

It is of interest to calculate the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear, with series expansion

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop coefficient. The one-loop coeff. $c_1 = 6C_f$ is scheme-independent, the c_ℓ with $\ell \geq 2$ are scheme-dependent and have been calculated up to 4-loop level in $\overline{MS}$ scheme, as noted above.

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

In the IR chirally symmetric phase, an all-order calculation of γ evaluated at an all-order calculation of α_{IR} would be an exact property of the theory.

In the χ bk. phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum behavior $\Sigma(k) \sim \Lambda(\Lambda/k)^{2-\gamma}$. In both phases, γ is bounded above as $\gamma < 2$.

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and fermions in the fundamental representation:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Plots of γ as fn. of N_f for SU(2) and SU(3):

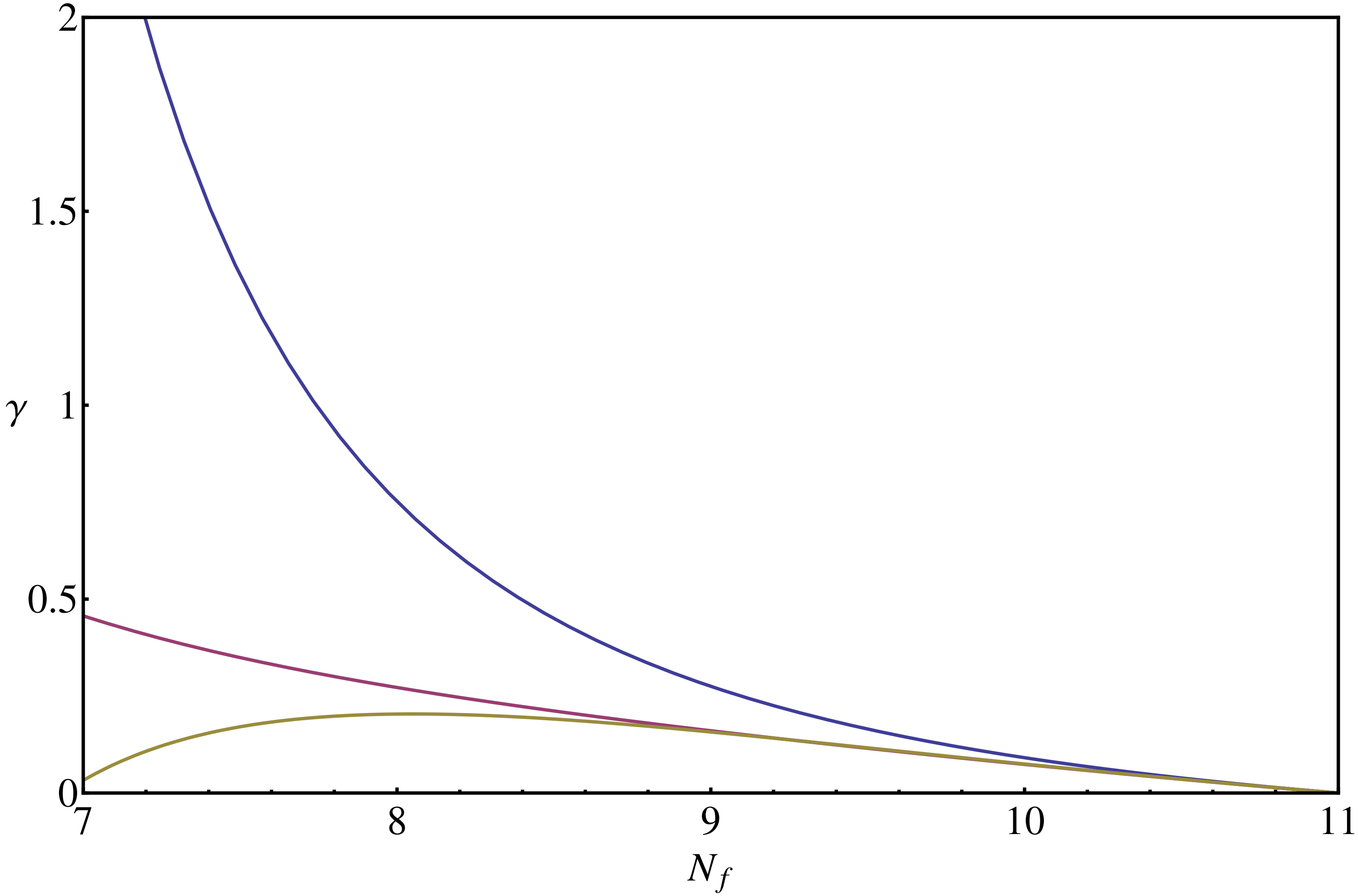

Figure 1: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(2) with $\boldsymbol{N_f}$ fermions in fund. rep. (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

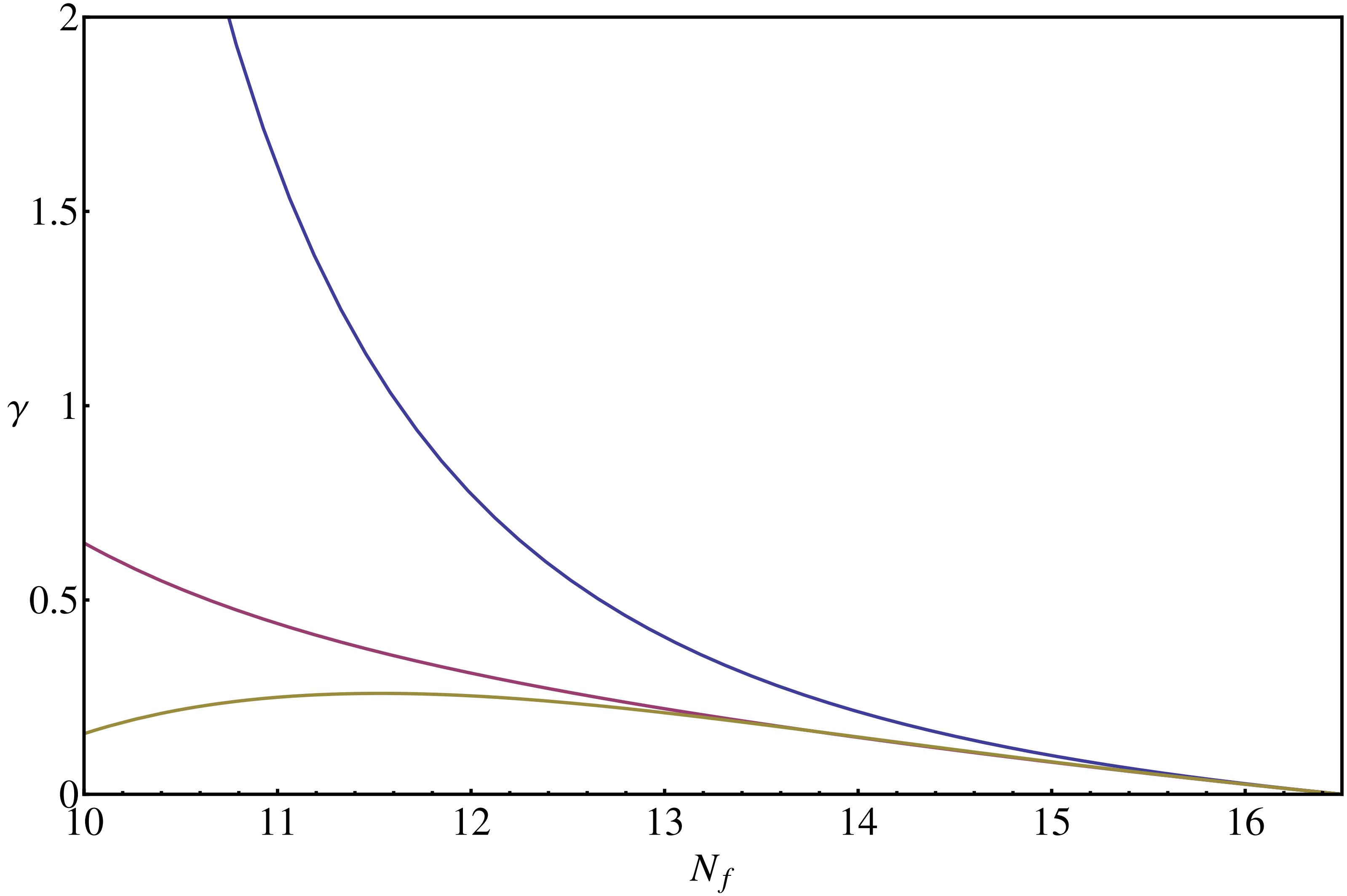

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(3) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

A necessary condition for a perturbative calculation to be reliable is that higher-order contributions do not modify the result too much. We find that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

We have also done higher-loop calcs. for a supersymmetric gauge theory in Ryttov and Shrock, Phys. Rev. D 85, 076009 (2012) (arXiv:1202.1297) - not discussed here.

So our higher-loop calcs. of α_{IR} and γ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings, closer to $N_{f,cr}$. Of course, perturbative calculations are not applicable when α is too large.

We have also performed these higher-loop calculations for larger fermion reps. R. In general, we find that, for a given N_c, R, and N_f, the values of $\gamma_{IR,n\ell}$ calculated to 3-loop and 4-loop order are smaller than the 2-loop value.

Example of a Comparison with Lattice Measurements

For SU(3) with $N_f = 12$, we calculate

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

some lattice results (N.B.: error estimates do not include all systematic uncertainties)

$\gamma = 0.414 \pm 0.016$ (Appelquist et al., PRD 84, 054501 (2011), IR χ-sym.)

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011), IR χ-sym.)

$0.2 \lesssim \gamma \lesssim 0.4$ (Kuti et al. (method-dep.) arXiv:1205.1878, arXiv:1211.3548, 1211.6164, PTP, finding SχSB)

$\gamma = 0.4 - 0.5$ (Y. Aoki et al., (LatKMI) PRD 86, 054506 (2012))

$\gamma = 0.27(3)$ (Hasenfratz et al., arXiv:1207.7162; $\gamma = 0.32(3)$, arXiv:1301.1355, IR χ-sym.)

So here the 2-loop value is larger than, and the 3-loop and 4-loop values closer to, these lattice measurements. Thus, our higher-loop calculations of γ yield better agreement with these lattice measurements than two-loop calculations.

Further Higher-Loop Structural Properties of β

In addition to $\alpha_{IR,n\ell}$, further interesting structural properties of the n-loop beta fn. $\beta_{n\ell}$ include

- the derivative $\beta'_{IR,n\ell} \equiv \frac{d\beta_{n\ell}}{d\alpha}$ evaluated at $\alpha_{IR,n\ell}$.
- the magnitude and location of the minimum in $\beta_{n\ell}$

In quasi-scale-invariant case where $\alpha_{IR} \gtrsim \alpha_{cr}$, dilaton mass relevant in dynamical EWSB models depends on how small β is for α near to α_{IR} and hence, at n-loop order, on $\beta'_{IR,n\ell}$, via the series expansion of $\beta_{n\ell}$ around $\alpha_{IR,n\ell}$,

$$\beta_{n\ell}(\alpha) = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

We have calculated these structural properties analytically and numerically (RS, PRD 87, 105005 (2013), arXiv:1301.3209.

Derivative of 2-loop β function at $\alpha_{IR,2\ell}$:

$$\beta'_{IR,2\ell} = -\frac{2b_1^2}{b_2} = \frac{2b_1^2}{|b_2|} = \frac{2(11C_A - 4T_f N_f)^2}{3[4(5C_A + 3C_f)T_f N_f - 34C_A^2]}$$

At 3-loop level:

$$\beta'_{IR,3\ell} = \frac{1}{|b_3|^2}\Big[-4|b_2|(b_2^2 + b_1|b_3|) + (b_2^2 + 2b_1|b_3|)\sqrt{b_2^2 + 4b_1|b_3|} \ \Big]$$

We prove a general inequality: for a given gauge group G, fermion rep. R, and $N_f \in I$ (in a scheme with $b_3 < 0$, which thus preserves the existence of the 2-loop IR zero in β at 3-loop level),

$$\beta'_{IR,3\ell} < \beta'_{IR,2\ell}$$

We carry out a similar analysis of the derivative of the 4-loop β function evaluated at $\alpha_{IR,4\ell}$, denoted $\beta'_{IR,4\ell}$, and find a similar decrease from 3-loop to 4-loop order. Some numerical values:

N_c	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220

Illustrative figures for SU(2) with $N_f = 8$ fermions and SU(3) with $N_f = 12$ fermions:

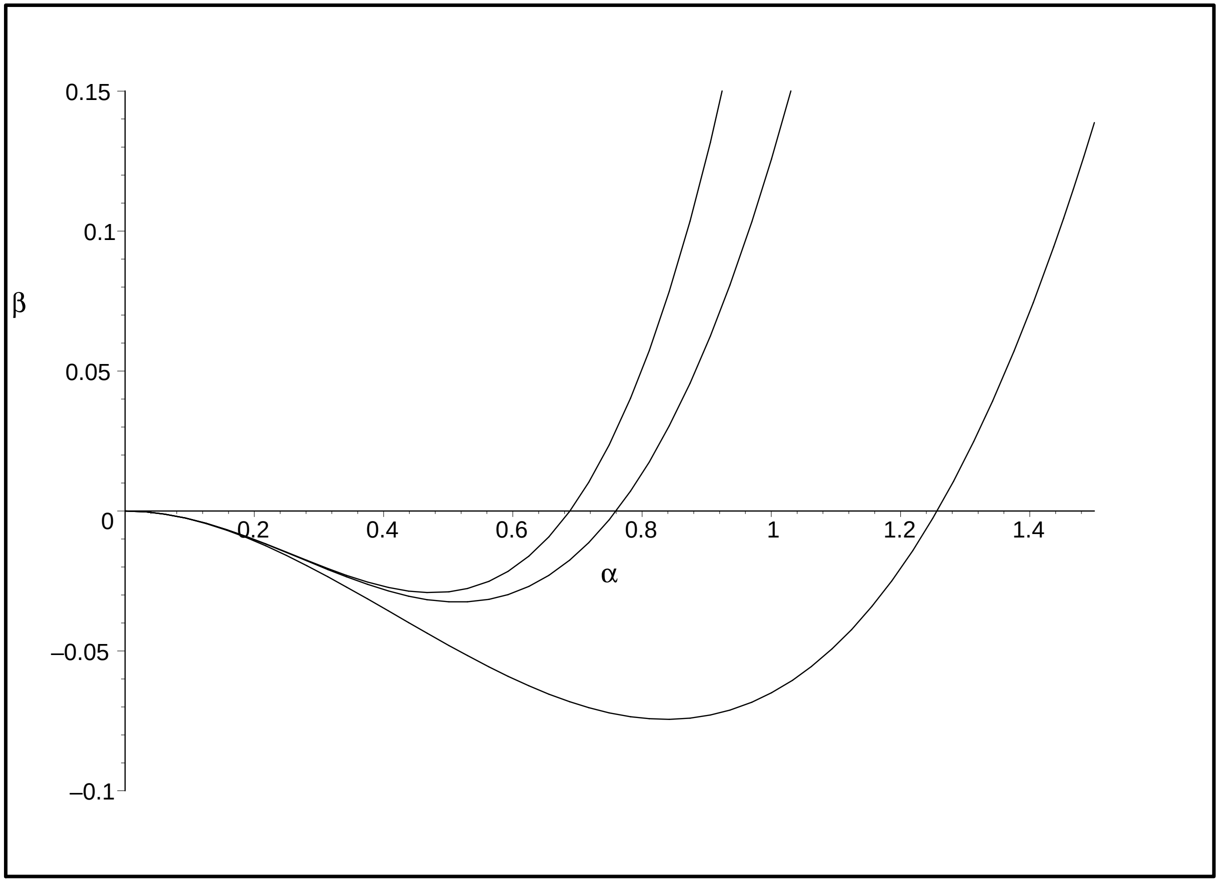

Figure 3: $\boldsymbol{\beta_{n\ell}}$ for SU(2), $\boldsymbol{N_f = 8}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

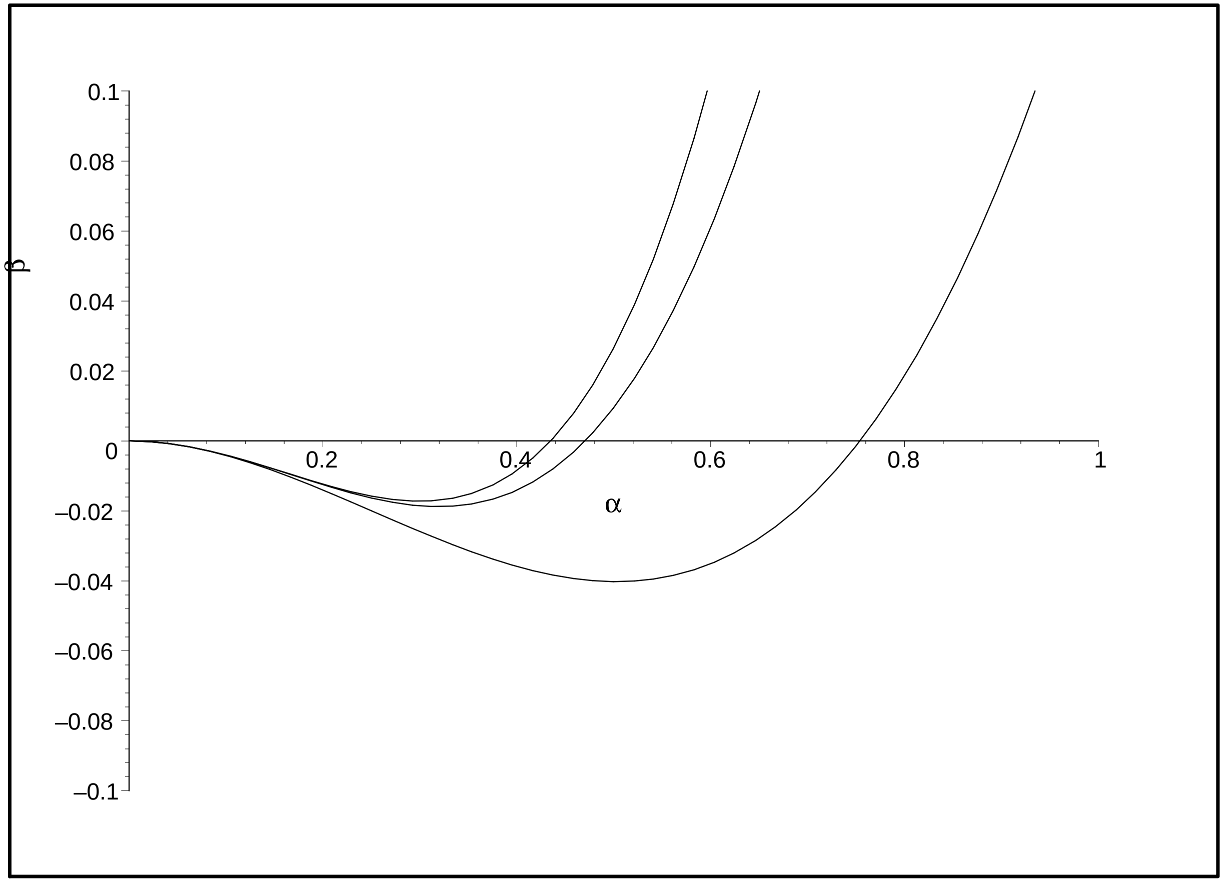

Figure 4: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell} N_c$, $\gamma_{IR,n\ell}$, and other structural properties of $\beta_{n\ell}$ are similar in theories with different values of N_c and N_f if they have equal or similar values of $r = N_f/N_c$.

This motivates a study of the UV to IR evolution of an SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the 't Hooft-Veneziano limit $N_c \to \infty$, $N_f \to \infty$ with

$$r \equiv \frac{N_f}{N_c} \quad \text{fixed}, \quad \alpha(\mu) N_c \equiv \xi(\mu) \quad \text{indep. of } N_c$$

Denote this as the LNN (large N_c, large N_f) limit. Asymptotic freedom requires $r < 11/2$. $\beta_{\xi,2\ell}$ has IR zero for $34/13 < r < 11/2$, i.e., $2.62 < r < 5.5$.

We have carried out this study in RS, PRD 87, 116007 (2013), arXiv:1302.5434. Our results provide a unified quantitative understanding of the similarities in UV to IR evolution of SU(N_c) theories with different N_c and N_f but similar r.

With $\xi = \alpha N_c$ and $x = a N_c = \xi/(4\pi)$, define a rescaled beta function that is finite in the LNN limit:

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{LNN} \beta_\alpha N_c$$

Denote the IR zero of n-loop β_ξ as $\xi_{IR,n\ell}$. By same type of analysis as before, we find

$$\xi_{IR,3\ell} \le \xi_{IR,2\ell}$$

$$\xi_{IR,4\ell} < \xi_{IR,3\ell} \quad \text{if} \quad 2.615 < r < 3.119$$

$$\xi_{IR,4\ell} > \xi_{IR,3\ell} \quad \text{if} \quad 3.119 < r < 5.500$$

Numerical values given in next table. The magnitude of the fractional difference

$$\frac{|\xi_{IR,4\ell} - \xi_{IR,3\ell}|}{\xi_{IR,4\ell}}$$

is reasonably small.

r	$\xi_{IR,2\ell}$	$\xi_{IR,3\ell}$	$\xi_{IR,4\ell}$
2.8	28.274	3.573	3.323
3.0	12.566	2.938	2.868
3.2	7.606	2.458	2.494
3.4	5.174	2.076	2.168
3.6	3.731	1.759	1.873
3.8	2.774	1.489	1.601
4.0	2.095	1.252	1.349
4.2	1.586	1.041	1.115
4.4	1.192	0.8490	0.9003
4.6	0.8767	0.6725	0.7038
4.8	0.6195	0.5083	0.5244
5.0	0.4054	0.3538	0.3603
5.2	0.2244	0.2074	0.2089
5.4	0.06943	0.06769	0.06775

We also study the anomalous dimension $\gamma_m \equiv \gamma$ of $\bar{\psi}\psi$ in this LNN limit.

Denote the n-loop γ evaluated at n-loop IR zero of β_ξ as $\gamma_{IR,n\ell}$.

e.g., at 2-loop level,

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

with similar results for higher-loop order.

Numerical values:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

General inequalities as before: $\gamma_{IR,3\ell} < \gamma_{IR,2\ell}$, $\gamma_{IR,4\ell} < \gamma_{IR,2\ell}$

We have studied the approach to the LNN limit and find that this is quite rapid, with leading correction terms suppressed by $1/N_c^2$. For example,

$$\alpha_{IR,2\ell} N_c = \frac{4\pi(11-2r)}{13r-34} + \frac{12\pi r(11-2r)}{(34-13r)^2 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$
$$+\frac{(11-2r)(18836-5331r+648r^2-140r^3)}{(13r-34)^3 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

This explains the approximate universality that is exhibited in calculations of these quantities for different (finite) values of N_c and N_f with similar or identical values of r.

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence.

Extensive studies of scheme dependence in QCD, relevant for high-energy quark-parton processes where $\alpha_s(\mu)$ is small, governed by the UV fixed point (Brodsky, Lepage, MacKenzie; Celmaster and Gonsalves; Stevenson; Garkusha, Gorishny, Kataev, Larin, Surguladze; Gracey; Brodsky, Mojaza, Wu...)

Here we focus not on theories such as QCD near the UV fixed point but on scheme dependence in calculation of an approx. or exact infrared fixed point of an asymptotically free theory: results in Ryttov and Shrock, PRD 86, 065032 (2012), arXiv:1206.2366; PRD 86, 085005 (2012), arXiv:1206.6895; and Shrock, PRD 88, 036003 (2013), arXiv:1305.6524.

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ to keep UV properties unchanged. Write

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s \ ,$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

The Jacobian $J = da/da' = d\alpha/d\alpha' = 1 + \sum_{s=1}^{s_{max}} (s+1) k_s (a')^s$, satisfying $J = 1$ at $a = a' = 0$.

After scheme transformation is applied, beta function in new scheme is

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\beta_\alpha \ .$$

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell \ ,$$

where $\bar{b}'_\ell = b'_\ell/(4\pi)^\ell$.

We calculate the b'_ℓ as functions of the b_ℓ and k_s. At 1-loop and 2-loop, this yields the well-known results

$$b'_1 = b_1\ , \quad b'_2 = b_2$$

We find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1\ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$

$$+(4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

etc. at higher-loop order.

A physically acceptable ST must satisfy several conditions:

C_1: the ST must map a (real positive) α to a real positive α', since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex α' would violate the unitarity of the theory.

C_2: the ST should not map a moderate value of α, where perturbation theory is applicable, to a value of α' so large that pert. theory is inapplicable.

C_3: J should not vanish, or else there would be a pole in $\beta_{\alpha'}$

C_4: Existence of an IR zero of β is a scheme-independent property, so the ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These conditions can always be satisfied by an ST near the UVFP at $\alpha = \alpha' = 0$, but they are not automatic, and can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1+ra}{1-ra}\right)$$

This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable. For, say, $r = 8\pi$, this pathology can occur at the moderate value $\alpha = 0.5$.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example,

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1+(ra)^2}\,\right]$$

We find reasonably small scheme-dependence for moderate α_{IR}.

Since the b_n with $n \geq 3$ are scheme-dependent, one might expect that it would be possible, at least in the vicinity of the UVFP at $\alpha = \alpha' = 0$, to construct a scheme transformations that would set $b'_n = 0$ for some range of $n \geq 3$, and, indeed a ST that would do this for all $n \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed an explicit scheme transformation that does this in the vicinity of the UVFP at $\alpha = \alpha' = 0$.

To construct this ST, first, solve eq. $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1} + \frac{b_2}{b_1} k_1 + k_1^2$$

Next, substitute this into expression for b'_4 and solve eq. $b'_4 = 0$, obtaining

$$k_3 = \frac{b_4}{2b_1} + \frac{3b_3}{b_1} k_1 + \frac{5b_2}{2b_1} k_1^2 + k_1^3$$

Continue this procedure iteratively.

Our studies give a quantitative assessment of the scheme dependence of an IR zero of β at loop order $n \geq 3$.

Conclusions

- Neutrino masses and lepton mixing are of great significance; neutrino electromagnetic properties can also be important.

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental interest and can be relevant to exploring BSM physics.

- Our higher-loop calculations give information on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$, provide comparison with lattice measurements.

- Results on the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed yield understanding of similarities in UV to IR flows in theories with different N_c and N_f but similar r.

- We have investigated effects of scheme-dependence of IR zero in higher-loop calculations and have pointed out that scheme transformations are subject to conditions that are easily satisfied at a UVFP but are a significant constraint at an IRFP.