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The renormalization scheme producing the NSVZ beta-function for $\mathcal{N}=1$ SQED regularized by higher derivatives

NSVZ β-function

The β-function in supersymmetric theories is related with the anomalous dimensions of the matter superfields via the relation

$$\beta(\alpha) = -\frac{\alpha^2\Big[3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big]}{2\pi(1 - C_2\alpha/2\pi)}.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. B 229, (1983), 381; Phys.Lett. 166B, (1985), 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. B 277, (1986), 456.

For the $\mathcal{N}=1$ supesymmetric electrodynamics (SQED) the NSVZ β-function has the form

$$\beta(\alpha) = \frac{\alpha^2}{\pi}\Big(1 - \gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. 42, (1985), 224; Phys.Lett. 166B, (1986), 334.

With the dimensional reduction in the $\overline{MS}$-scheme it agrees with the explicit calculations in only the two-loop approximation.

$N=1$ SQED, regularized by higher derivatives

In order to regularize the $N=1$ SQED by higher derivatives it is necessary to add the higher derivative term to the action:

A.A.Slavnov, Nucl.Phys., **B31**, (1971), 301; Theor.Math.Phys. **13**, (1972), 1064.
V.K.Krivoshchekov, Theor.Math.Phys. **36**, (1978), 745; P.West, Nucl.Phys. B268, (1986), 113.

$$S_{\text{reg}} = \frac{1}{4e^2}\text{Re}\int d^4x\,d^2\theta\,W^a R(\partial^2/\Lambda^2)W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi^* e^{2V}\phi + \widetilde{\phi}^* e^{-2V}\widetilde{\phi}\Big)$$

where $R(\partial^2/\Lambda^2)$ is a regulator, e.g. $R = 1+\partial^{2n}/\Lambda^{2n}$. Here ϕ_i and $\widetilde{\phi}$ are chiral matter superfields, V is a real gauge superfield, and $W_a = \bar{D}^2 D_a V/4$.

Adding the higher derivative term allow to remove all divergences beyond the one-loop approximation. In order to remove them we insert in the generating functional the Pauli–Villars determinants:

$$Z[J,\mathbf{\Omega}] = \int D\mu \prod_I \Big(\det PV(V,M_I)\Big)^{c_I} \exp\Big\{iS_{\text{reg}} + iS_{\text{gf}} + \text{Sources}\Big\},$$

$\sum\limits_I c_I = 1$; $\sum\limits_I c_I M_I^2 = 0$; $M_I = a_I\Lambda$. (Λ is the only dimensionful parameter.)

Renormalization

$$\Gamma^{(2)} = \int \frac{d^4p}{(2\pi)^4}\, d^4\theta \Big(-\frac{1}{16\pi} V(-p)\, \partial^2\Pi_{1/2} V(p)\, d^{-1}(\alpha_0, \Lambda/p) + $$
$$+\frac{1}{4}\Big(\phi^*(-p,\theta)\,\phi(p,\theta) + \widetilde{\phi}^*(-p,\theta)\,\widetilde{\phi}(p,\theta)\Big)\, G(\alpha_0,\Lambda/p)\Big).$$

where $\partial^2\Pi_{1/2}$ is a supersymmetric transversal projection operator.

Then we defined the renormalized coupling constant $\alpha(\alpha_0,\Lambda/\mu)$, requiring that the inverse invariant charge $d^{-1}(\alpha_0(\alpha,\Lambda/\mu),\Lambda/p)$ is finite in the limit $\Lambda\to\infty$.

The renormalization constant Z_3 is defined by

$$\frac{1}{\alpha_0} \equiv \frac{Z_3(\alpha,\Lambda/\mu)}{\alpha}.$$

The renormalization constant Z is constructed, requiring that the renormalized two-point Green function ZG is finite in the limit $\Lambda\to\infty$:

$$G_{\text{ren}}(\alpha,\mu/p) = \lim_{\Lambda\to\infty} Z(\alpha,\Lambda/\mu) G(\alpha_0,\Lambda/p).$$

Renormgroup functions defined in terms of the bare coupling constant and NSVZ relation

$$\beta\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv \frac{d\alpha_0(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}};$$

$$\gamma\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv -\frac{d\ln Z(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}},$$

These RG functions do not depend on the renormalization prescription.

Due to the factorization of integrands into total derivatives using the higher derivative regularization it is possible to obtain

K.S., Nucl.Phys. B **852** (2011) 71.

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0}$$

$$= \frac{1}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big) = \frac{1}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$

Therefore, the NSVZ relation is naturally obtained for the RG functions defined in terms of the bare coupling constant.

Three-loop calculation for $\mathcal{N}=1$ SQED

$$
\begin{aligned}
\frac{\beta(\alpha_0)}{\alpha_0^2} &= 2\pi\frac{d}{d\ln\Lambda}\Bigg\{\sum_I c_I\int\frac{d^4q}{(2\pi)^4}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\frac{\ln(q^2+M^2)}{q^2}+4\pi\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{e^2}{k^2R_k^2}\\
&\times\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\left(\frac{1}{q^2(k+q)^2}-\sum_I c_I\frac{1}{(q^2+M_I^2)((k+q)^2+M_I^2)}\right)\left[R_k\left(1+\frac{e^2}{4\pi^2}\ln\frac{\Lambda}{\mu}\right)\right.\\
&\left.-2e^2\left(\int\frac{d^4t}{(2\pi)^4}\frac{1}{t^2(k+t)^2}-\sum_J c_J\int\frac{d^4t}{(2\pi)^4}\frac{1}{(t^2+M_J^2)((k+t)^2+M_J^2)}\right)\right]\\
&+4\pi\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{e^4}{k^2R_kl^2R_l}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Bigg\{\Bigg(-\frac{2k^2}{q^2(q+k)^2(q+l)^2(q+k+l)^2}\\
&+\frac{2}{q^2(q+k)^2(q+l)^2}\Bigg)-\sum_I c_I\Bigg(-\frac{2(k^2+M_I^2)}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)}\\
&\times\frac{1}{((q+k+l)^2+M_I^2)}+\frac{2}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)}-\frac{1}{(q^2+M_I^2)^2}\\
&\times\frac{4M_I^2}{((q+k)^2+M_I^2)((q+l)^2+M_I^2)}\Bigg)\Bigg\}
\end{aligned}
$$

The RG functions defined in terms of the renormalized coupling constant and the NSVZ scheme

Usually the RG functions are defined in terms of the renormalized coupling constant:

$$\widetilde{\beta}\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\alpha(\alpha_0,\Lambda/\mu)}{d\ln\mu}\bigg|_{\alpha_0=\mathrm{const}};$$

$$\widetilde{\gamma}\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d}{d\ln\mu}\ln ZG(\alpha_0,\Lambda/\mu)\bigg|_{\alpha_0=\mathrm{const}}$$

$$= \frac{d\ln Z(\alpha(\alpha_0,\Lambda/\mu),\Lambda/\mu)}{d\ln\mu}\bigg|_{\alpha_0=\mathrm{const}},$$

Unlike the RG functions β and γ (defined in terms of the bare coupling constant) these RG functions are scheme-dependent. This means that they depend on the arbitrariness of choosing α and Z.

For these RG functions the NSVZ relation is valid only in a certain subtraction scheme, called the NSVZ scheme.

The NSVZ scheme with the higher derivative regularization

If $\mathcal{N}=1$ SQED is regularized by higher derivatives, the NSVZ scheme can be constructed in all loops by imposing an additional condition on the renormalization constants, which can be formulated as follows: There should be a point $x_0 = \ln\Lambda/\mu_0$ such that

$$\alpha_0(\alpha_{\text{NSVZ}}, x_0) = \alpha_{\text{NSVZ}}; \qquad Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, x_0) = 1$$

for all values of α_{NSVZ}. Equivalently, there is a point x_0 such that

$$(Z_3)_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, x_0) = 1; \qquad Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, x_0) = 1.$$

(If the theory is regularized by the dimensional reduction, there is no similar condition.) For proving this statement we note that the above conditions give

A.K.,K.S., Nucl.Phys. B **875** (2013) 459 (arXiv:1305.7094 [hep-th]).

$$\widetilde{\beta}(\alpha_0) = \beta(\alpha_0); \qquad \widetilde{\gamma}(\alpha_0) = \gamma(\alpha_0),$$

and the RG functions β and γ defined in terms of the bare coupling constant satisfy NSVZ relation.

The scheme dependence in the three-loop approximation

The two-loop Green function of the matter superfields is given by

$$G(\alpha_0,\Lambda/p) = 1 - \int \frac{d^4k}{(2\pi)^4}\frac{2e_0^2}{k^2 R_k (k+p)^2} + \int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4e_0^4}{k^2R_k l^2 R_l}$$

$$\times\left(\frac{1}{(k+p)^2(l+p)^2} + \frac{1}{(l+p)^2(k+l+p)^2} - \frac{(k+l+2p)^2}{(k+p)^2(l+p)^2(k+l+p)^2}\right)$$

$$+\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{4e_0^4}{k^2R_k^2(k+p)^2}\left(\frac{1}{l^2(k+l)^2} - \sum_{I=1}^{n} c_I \frac{1}{(l^2+M_I^2)\left((k+l)^2+M_I^2\right)}\right)$$

$$+O(e_0^6),$$

For $R_k = 1 + k^{2n}/\Lambda^{2n}$ it is possible to find a divergent part of this expression and the corresponding renormalization constant. Then the three-loop renormalization of the coupling constant can be found using the relation

$$\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} = \frac{1}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big).$$

The scheme dependence in the three-loop approximation

The (three-loop) result for the renormalized coupling constant is not uniquely defined:

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{1}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) - \frac{\alpha}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big)$$

$$-\frac{\alpha^2}{\pi^3}\Big(\frac{1}{2}\ln^2\frac{\Lambda}{\mu} - \ln\frac{\Lambda}{\mu}\sum_{I=1}^{n} c_I \ln a_I - \frac{3}{2}\ln\frac{\Lambda}{\mu} + b_1\ln\frac{\Lambda}{\mu} + b_3\Big) + O(\alpha^3),$$

where b_i are arbitrary finite constants.

Similarly, the renormalization constant Z (in the two-loop approximation) for the matter superfields is not also uniquely defined:

$$Z = 1 + \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big)$$

$$+\frac{\alpha^2}{\pi^2}\ln^2\frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2}\ln\frac{\Lambda}{\mu}\Big(\sum_{I=1}^{n} c_I \ln a_I - b_1 + \frac{3}{2} - g_1\Big) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3),$$

where g_i are other arbitrary finite constants.

The subtraction scheme is fixed by fixing values of the constants b_i and g_i.

The scheme dependence in the three-loop approximation

The RG functions defined in terms of the bare coupling constant are

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{1}{\pi} + \frac{\alpha_0}{\pi^2} - \frac{\alpha_0^2}{\pi^3}\Big(\sum_{I=1}^{n} c_I \ln a_I + \frac{3}{2}\Big) + O(\alpha_0^3);$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2}\Big(\frac{3}{2} + \sum_{I=1}^{n} c_I \ln a_I\Big) + O(\alpha_0^3).$$

They do not depend on the finite constants b_i and g_i (i.e. they are scheme-independent) and satisfy the NSVZ relation.

The RG functions defined in terms of the renormalized coupling constant are

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{1}{\pi} + \frac{\alpha}{\pi^2} - \frac{\alpha^2}{\pi^3}\Big(\sum_{I=1}^{n} c_I \ln a_I + \frac{3}{2} - b_1 + b_2\Big) + O(\alpha^3);$$

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{3}{2} + \sum_{I=1}^{n} c_I \ln a_I - b_1 + g_1\Big) + O(\alpha^3)$$

and depend on a subtraction scheme.

The NSVZ scheme in the three-loop approximation

The NSVZ scheme is determined by the conditions

$$\alpha_0(\alpha_{\textsf{NSVZ}}, x_0) = \alpha_{\textsf{NSVZ}}; \qquad Z_{\textsf{NSVZ}}(\alpha_{\textsf{NSVZ}}, x_0) = 1$$

For simplicity we set $g_1 = 0$ (this constant can be excluded by a redefinition of μ). In this case $x_0 = 0$ and the above conditions (for the NSVZ scheme) give

$$g_2 = b_1 = b_2 = b_3 = 0.$$

In this case in the considered approximations

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{1}{\pi} + \frac{\alpha}{\pi^2} - \frac{\alpha^2}{\pi^3}\Big(\sum_{I=1}^{n} c_I \ln a_I + \frac{3}{2}\Big) + O(\alpha^3) = \frac{\beta(\alpha)}{\alpha^2};$$

$$\widetilde{\gamma}(\alpha) = \frac{d\ln Z}{d\ln\mu} = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{3}{2} + \sum_{I=1}^{n} c_I \ln a_I\Big) + O(\alpha^3) = \gamma(\alpha).$$

As a consequence, in this scheme the NSVZ relation is satisfied.

Finite renormalizations

Under a finite renormalization

$$\alpha \to \alpha'(\alpha); \qquad Z'(\alpha', \Lambda/\mu) = z(\alpha) Z(\alpha, \Lambda/\mu)$$

the β-function and the anomalous dimension defined in terms of the renormalized coupling constant are changed according to the following rules:

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\ln\mu}\Big|_{\alpha_0=\mathrm{const}} = \frac{d\alpha'}{d\alpha}\widetilde{\beta}(\alpha);$$

$$\widetilde{\gamma}'(\alpha') = \frac{d\ln Z'}{d\ln\mu}\Big|_{\alpha_0=\mathrm{const}} = \frac{d\ln z}{d\alpha}\cdot\widetilde{\beta}(\alpha) + \widetilde{\gamma}(\alpha).$$

Using these equations it is easy to see that if $\widetilde{\beta}(\alpha)$ and $\widetilde{\gamma}(\alpha)$ satisfy the NSVZ relation, then

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\cdot\frac{\alpha^2}{\pi}\frac{1-\widetilde{\gamma}'(\alpha')}{1-\alpha^2(d\ln z/d\alpha)/\pi}\Big|_{\alpha=\alpha(\alpha')}.$$

Relation between the NSVZ and $\overline{\textbf{DR}}$ schemes in the three-loop approximation

Using the dimensional reduction and the $\overline{\mbox{DR}}$ scheme the three-loop β-function and the two-loop anomalous dimension for $\mathcal{N}=1$ SUSY theories was found in

I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138.

In particular, for $\mathcal{N}=1$ SQED

$$\widetilde\beta_{\overline{\mbox{DR}}}(\alpha) = \frac{\alpha^2}{\pi} + \frac{\alpha^3}{\pi^2} - \frac{5\alpha^4}{4\pi^3} + O(\alpha^5);$$

$$\widetilde\gamma_{\overline{\mbox{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} + O(\alpha^3).$$

These functions do not satisfy the NSVZ relation. The NSVZ scheme can be obtained after a finite renormalization

$$\alpha_{\overline{\mbox{DR}}} = \alpha_{\mbox{NSVZ}} - \frac{\alpha_{\mbox{NSVZ}}^3}{4\pi^2} + O(\alpha^4),$$

implicitly assuming that $Z_{\overline{\mbox{DR}}}(\alpha_{\overline{\mbox{DR}}},\Lambda/\mu) = Z_{\mbox{NSVZ}}(\alpha_{\mbox{NSVZ}},\Lambda/\mu)$.

Relation between the NSVZ and $\overline{\text{DR}}$ schemes in the three-loop approximation

If the NSVZ scheme is obtained using the higher derivative regularization, it is easy to see that

$$\frac{Z_{\overline{\text{DR}}}(\alpha_{\overline{\text{DR}}},\Lambda/\mu)}{Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}},\Lambda/\mu)} = 1 - \frac{\alpha_{\text{NSVZ}}}{\pi}\Big(\sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2} - b_1\Big) + O(\alpha_{\text{NSVZ}}^2);$$

$$\frac{1}{\alpha_{\overline{\text{DR}}}} = \frac{1}{\alpha_{\text{NSVZ}}} + \frac{b_1}{\pi} + \frac{\alpha_{\text{NSVZ}}}{4\pi^2} - \frac{\alpha_{\text{NSVZ}}}{\pi^2}\Big(\sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2} - b_1\Big) + O(\alpha_{\text{NSVZ}}^2),$$

where b_1 is an arbitrary finite constant. For

$$b_1 = \sum_{I=1}^{n} c_I \ln a_I + \frac{1}{2}$$

this result corresponds to the one obtained by I.Jack, D.R.T.Jones, and C.G.North. However, one more parameter should be fixed in order that the anomalous dimensions coincide.

Conclusion

- ✓ For $\mathcal{N}=1$ SQED regularized by higher derivatives the NSVZ β-function is naturally obtained for the renormgroup functions defined in terms of the bare coupling constant. The renormgroup functions defined in this way do not depend on the renormalization prescription.
- ✓ The NSVZ β-function appears because integrals which determine the β-function defined in terms of the bare coupling constant are factorized into integrals of double total derivatives.
- ✓ If the renormgroup functions are defined (in the standard way) in terms of the renormalized coupling constant, the NSVZ β-function is obtained in a special subtraction scheme, called the NSVZ scheme.
- ✓ In case of using the higher derivative regularization the NSVZ scheme for $\mathcal{N}=1$ SQED is defined by the boundary conditions $(Z_3)_{\mbox{NSVZ}}(\alpha_{\mbox{NSVZ}}, x_0)=1$ and $Z_{\mbox{NSVZ}}(\alpha_{\mbox{NSVZ}}, x_0)=1$ for the renormalization constants. It is related with the MOM scheme by finite renormalizations of the couling constant and the matter superfields.

Thank you for the attention!