Planar Undulator Performance and Harmonic Generation in a Constant Magnetic Field

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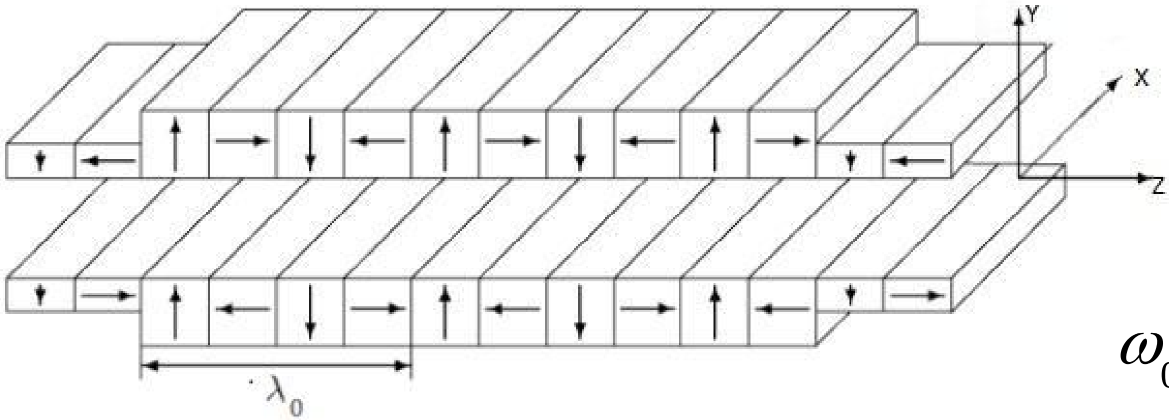
Undulator Radiation

- Properties of UR — high beam intensity, narrow cone of emission — decisive for UR applications. They gave rise and development to free electron lasers – FEL.
- Modern undulators produce fast and high coherent e^- beams, have high brightness, high ω — Roentgen range.
- They work with multiple magnetic fields and contain many periods to achieve given emission characteristics.
- Problems (some of them): distortions of the magnetic field, inhomogeneity of periodic structure, energy spread and beam emittance can effect operation of the devices significantly.
- Mathematical instruments for our analysis — modified special functions to obtain analytical expressions.

Planar Undulator with Transversal Constant Magnetic Component

- Magnetic field of a planar undulator, distorted by a constant magnetic field, usually present in undulators with constant magnets. Consider it in transversal plane:



$$B_d = B_0\kappa_1 = B_0\sqrt{\kappa^2 + \rho^2}$$

$$\vec{B} = B_0\left(\rho, \kappa + \sin(k_\lambda z), 0\right)$$

$$\omega_0 = k_\lambda \beta_z^0 c, \quad k_\lambda = (2\pi / \lambda_u),$$

- κ and ρ – coefficients, relating the amplitude of the constant constituent of the magnetic field B_d to the amplitude of the periodic magnetic field B_0.
- Common approximations for undulator problems:

$$\gamma >> 1, \; \beta_\perp << 1, \; \beta_\perp H_{||} << H_\perp, \; \vec{E} = 0$$

Radiation Integral

- Far zone: distance undulator – observer exceeds significantly undulator length. The radiation integral:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\left|\omega \int_{-\infty}^{\infty} dt\left[\vec{n}\times\left[\vec{n}\times\vec{\beta}\right]\right]\exp\left(i\omega\left(t-\vec{n}\vec{r}/c\right)\right)\right|^2$$

- Exact mathematical treatment: on-axis — generalized Bessel functions of two arguments: $J_l^{(m)}(u,v)$
- Angle distribution — generalized Bessel functions of four arguments. Generating exponent for them:

$$\sum_{k=-\infty}^{\infty} t^k J_k^{(m,p)}(x,y;u,v) =$$

$$\exp\left\{\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^m-\frac{1}{t^m}\right)+\frac{u}{2}\left(t^p-\frac{1}{t^p}\right)+\frac{v}{2}\left(t^{mp}-\frac{1}{t^{mp}}\right)\right\}$$

Add-On: Properties of Bessel Functions

- Integral presentation: $J_n^{(m,p)}(x,y;u,v) =$

$$\frac{1}{\pi}\int_0^{\pi} d\theta \cos[n\theta - x\sin(\theta) - y\sin(m\theta) - u\sin(p\theta) - v\sin(pm\theta)]$$

- Expansion in series of 2-variables generalized Bessel

$$J_n^{(m,p)}(x,y;u,v) = \sum_{l=-\infty}^{\infty} J_{n-pl}^{(m)}(x,y) J_l^{(m)}(u,v)$$

- Expansion in series of common Bessel functions.

$$J_n^{(m)}(x,y) = \sum_{l=-\infty}^{\infty} J_{n-ml}(x) J_l(y)$$

- Generating exponent: $\sum_{n=-\infty}^{\infty} e^{in\alpha} J_n^{(m,p)}(x,y,z,\rho) =$

$$\exp\{i[x\sin\alpha + y\sin m\alpha + z\sin p\alpha + \rho\sin pm\alpha]\}$$

- Symmetry properties — reflection property.

$$J_n^{(m,p)}(x,y,z,\rho) = (-1)^n J_{-n}^{(m,p)}(-x,-y,-z,-\rho)$$

Generalized Airy Functions

Non-periodic magnetic components are accounted for by generalized Airy functions.

Off the axis — $S(\alpha,\beta,\varepsilon)$ of three arguments :

$$S(\alpha, \beta, \varepsilon) \equiv \int_0^1 e^{i\left(\alpha\tau+\varepsilon\tau^2+\beta\tau^3\right)} d\tau,$$

On the undulator axis it reduces to $S(\alpha,\beta)$:

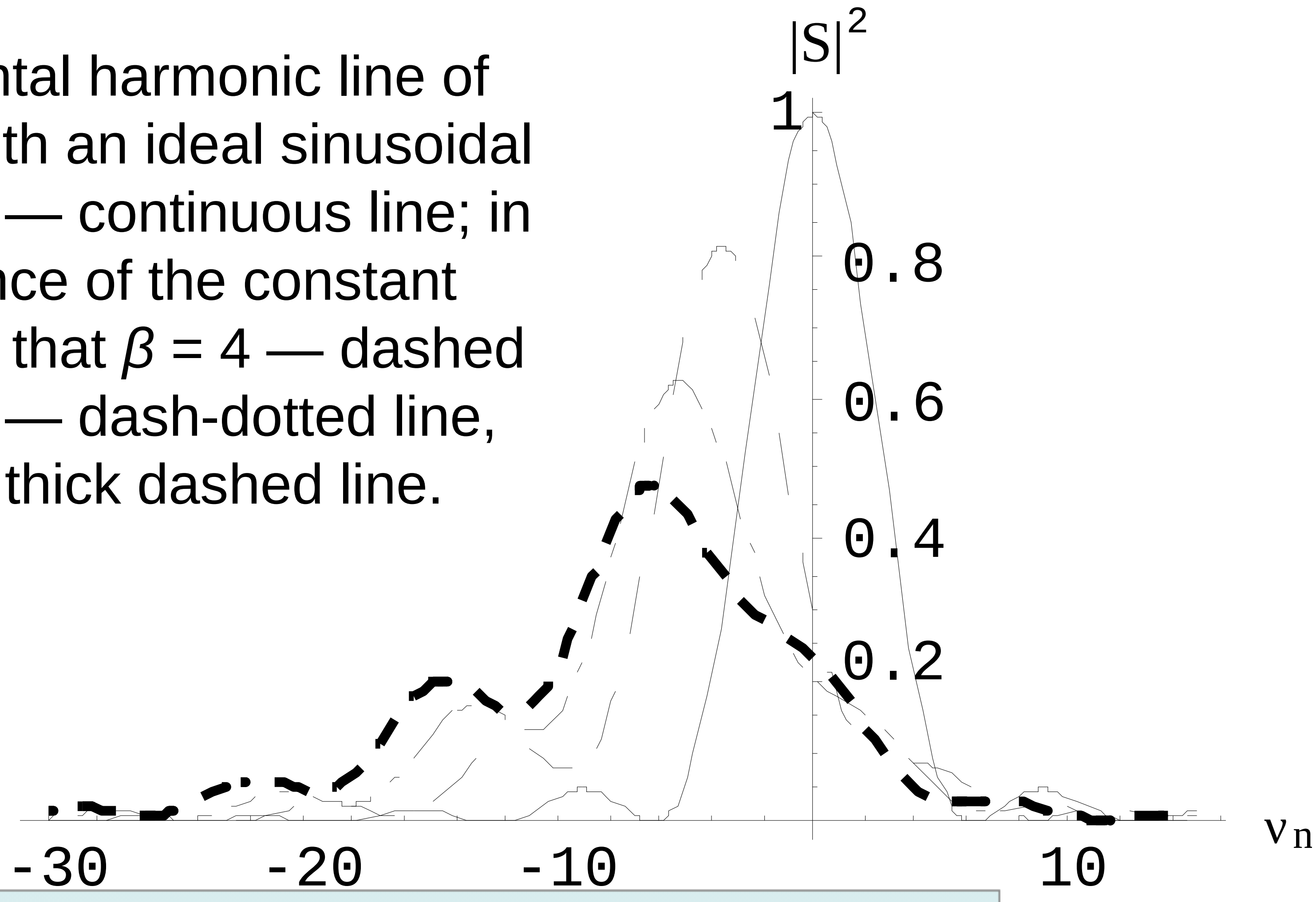
$$S(\alpha, \beta) = \int_0^1 e^{i\left(\alpha\tau+\beta\tau^3\right)} d\tau. = S(\alpha, \beta, 0)$$

In purely periodic oscillating field it becomes $\mathbf{sinc}(\alpha/2)$:

$$S(\alpha, 0, 0) = e^{i\frac{\alpha}{2}} \operatorname{sinc}\left(\frac{\alpha}{2}\right).$$

Fundamental UR harmonic

Fundamental harmonic line of the UR: with an ideal sinusoidal field $\beta = 0$ — continuous line; in the presence of the constant field, such that $\beta = 4$ — dashed line, $\beta = 6$ — dash-dotted line, $\beta = 7.5$ — thick dashed line.



UR line has good shape when $|\beta| < 8$

Common Planar Undulator spectrum

$$\omega_{n_0} = n\omega_{R_0} = \frac{2\omega_0 n\gamma^2}{1+k^2/2},$$

$k = eB_0\lambda_u/2\pi mc^2$ — undulator parameter,

n — harmonic number. $n = 0, 1, 2, 3\,etc.$

Undulator spectrum consists of the harmonics $n\omega_{R0}$ and it is characterized by the detuning parameter ν_n:

$$\nu_n = 2\pi Nn\left(\frac{\omega}{\omega_n} - 1\right),$$

ν_n is the α parameter in $S(\alpha,\beta)$.

ν_n = 0 determines the peak frequency of a common UR line.

Modified Undulator Spectrum

$$\omega_n\big|_{\psi=0} = n\omega_R = \frac{2n\gamma^2\omega_0}{1+\frac{k^2}{2}+\left(\gamma\vartheta_H\right)^2}, \; n = 0,1,2,3\,etc.$$

Bending angle
(like off the axis!):

$$\theta_H = \frac{2}{\sqrt{3}}\pi N\kappa_1\frac{k}{\gamma} = \frac{1}{\sqrt{3}\gamma}\left(\frac{e}{mc^2}\kappa_1 B_0 L\right)$$

- Bending angle ϑ_H, is due to constant constituent B_d.
- ϑ_H, depends on the absolute value of B_d field.
- Its effect is accumulated along the undulator and depends on the number of undulator periods N.
- The total undulator length L counts !

Spectrum Shift by a Constant Field

- Constant magnetic field shifts the spectrum with $\Delta\omega_n$:

$$\Delta\omega_R = \omega_n - \omega_{n0} = -\frac{\omega_{n0}}{1 + \dfrac{\left(1 + k^2/2\right)}{(\gamma\theta_H)^2}}$$

- Peak frequency of *n*-th harmonic of the undulator, affected by the constant magnetic field corresponds

$$\nu_{\mathrm{n\,Res}} = -\frac{2\pi N n\left(\gamma\theta_H\right)^2}{1 + k^2/2} = -\frac{n}{3}\frac{(2\pi N)^3\kappa_1^2}{1/2 + 1/k^2}$$

- It is exactly zero when $\kappa_1 = 0$ and since $\nu_{\mathrm{n\,Res}} \sim N^3 B_\mathrm{d}^2$ it is highly sensitive to the number of undulator periods and to the field intensity B_d.

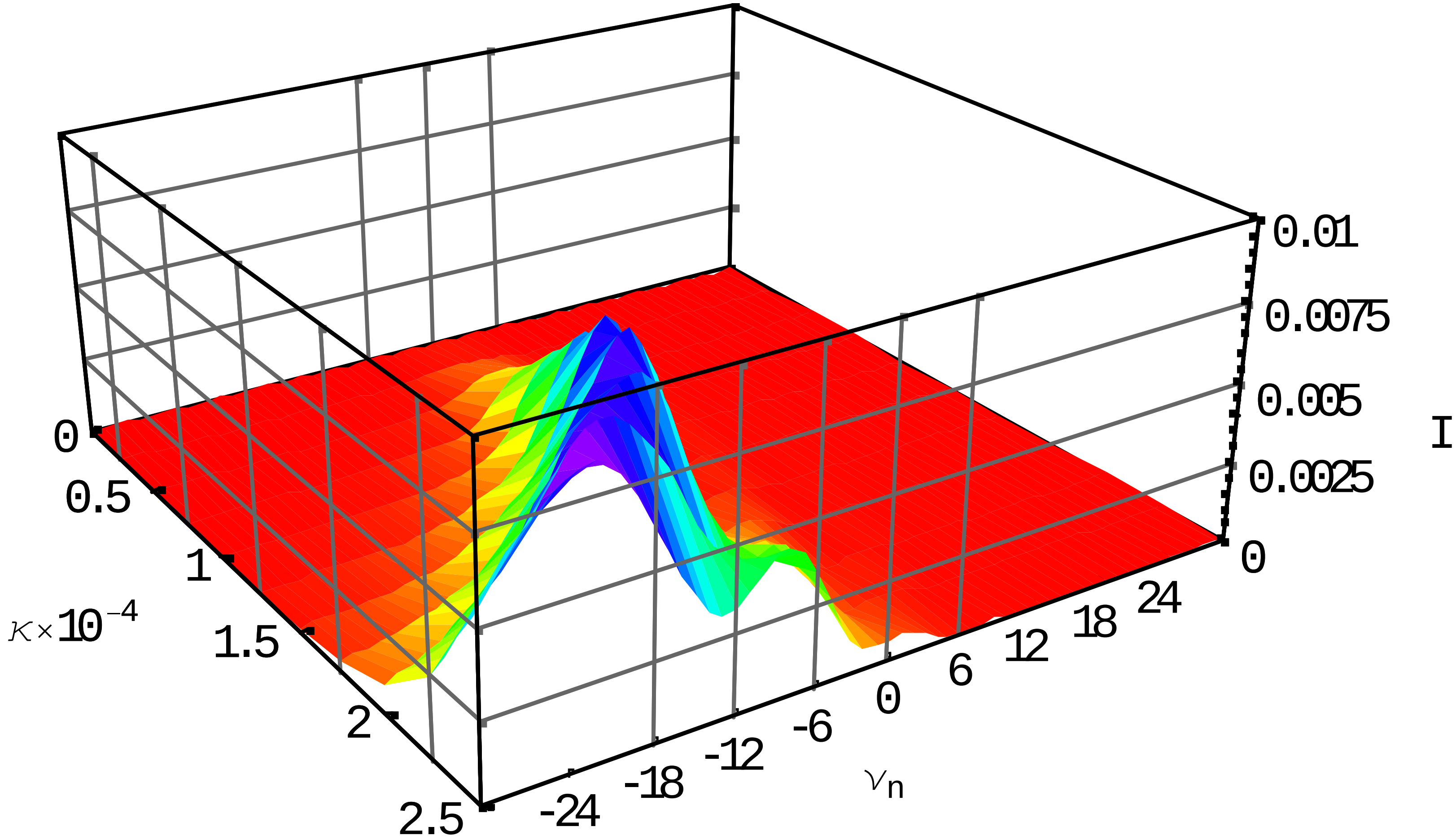
On-Axis UR intensity

- For the undulator radiation off the axis we have expressions, involving $\tilde{J}_{n,p}(x_0, x_1, x_2, x_3)$ and $S(\alpha, \beta, \varepsilon)$.
- On-axis we have 2-argument Bessel and Airy functions.
- In a weak B_d field ($\kappa,\rho \ll 1$), on-axis UR intensity becomes:

$$
\left.\frac{d^2 I}{d\omega d\Omega}\right|_{\psi=0} = \frac{e^2 N^2 \gamma^2}{c} \frac{k^2}{(1+k^2/2)^2} \times
$$

$$
\sum_{n=-\infty}^{\infty} n^2 \left\{ \left[S(\nu_n, \beta) \left(J_{\frac{n+1}{2}}\left(-\frac{\xi}{8}\right) + J_{\frac{n-1}{2}}\left(-\frac{\xi}{8}\right) \right) \right]^2 + (4\pi N \kappa_1)^2 \left[\frac{\partial S(\nu_n, \beta)}{\partial \nu_n} J_{\frac{n}{2}}\left(-\frac{\xi}{8}\right) \right]^2 \right\}
$$

On-Axis UR — Even (n=2) Harmonic

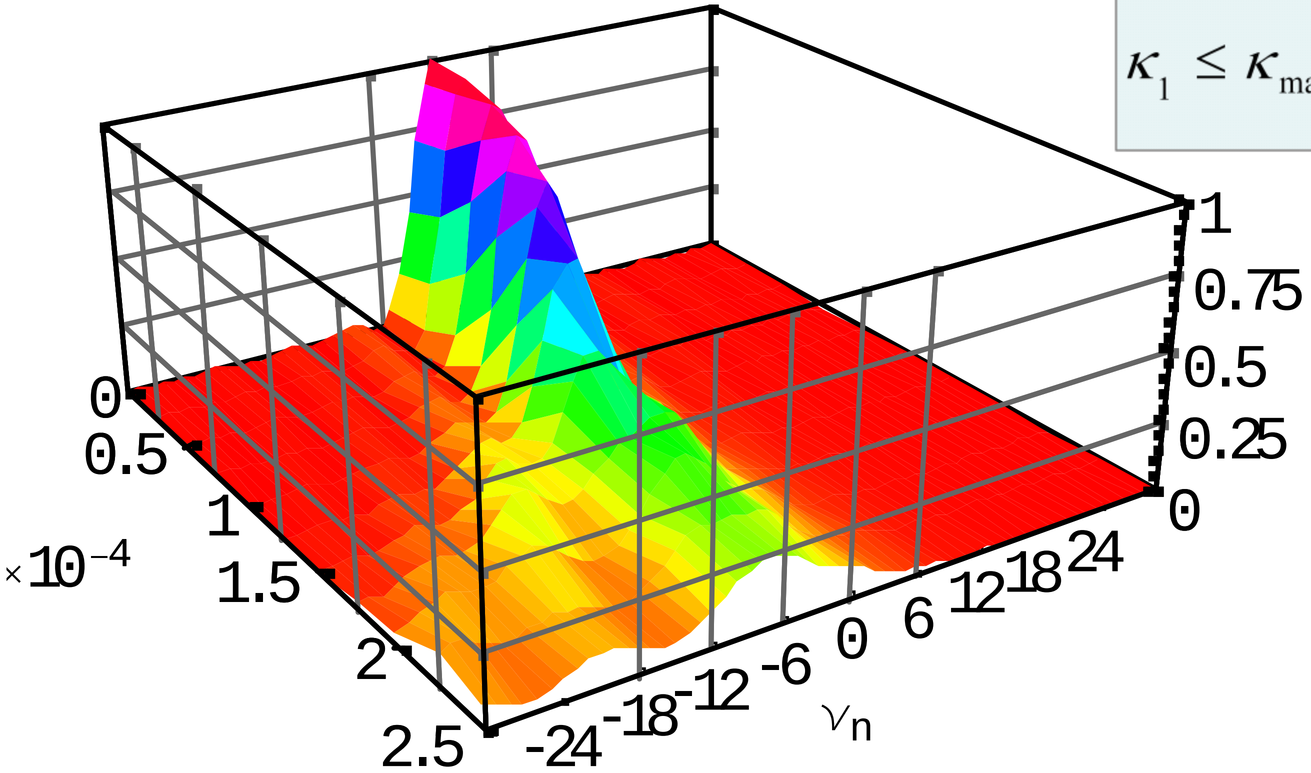


- The 2d harmonic of the undulator with $N = 100$, $k = 2$ due to the constant field $B_d = \kappa_1 B_0$. Note **low intensity**.

Condition for Good UR Line Shape

•Condition for good line shape of UR harmonic, in the presence of the constant magnetic field B_d:

$$\kappa_1 \leq \kappa_{\max} = \frac{1}{(\pi N)^{3/2}} \sqrt{\frac{3}{n}\left(\frac{1}{2} + \frac{1}{k^2}\right)}$$



Otherwise, we have a flat broad frequency spread of the radiation. $\nu_{n\,\mathrm{Res}}$ is in the middle of the UR frequency band — much wider than common harmonic.

$$\nu_n \in [0,\ 2\nu_{n\,\mathrm{Res}}]$$

$$\Delta\nu_{n0} \approx 2\pi$$

• 3d harmonic broadening, $N = 100$, $k = 2$.

Emission Line Width and Broadening by the Constant Field

•Half-width of the common UR harmonic:

$$\frac{\Delta\omega}{\omega_{n0}} = \frac{\omega - \omega_{n0}}{\omega_{n0}} = \frac{1}{nN} << 1$$

for any realistic N

Half-width of the broadening due to the constant magnetic field:

$$\left|\frac{\Delta\omega_R}{\omega_n}\right| = \frac{(\gamma\theta_H)^2}{1 + k^2/2} = \frac{1}{3}\frac{(2\pi N\kappa_1)^2}{1/2 + 1/k^2} << 1$$

Is surely fulfilled if $\kappa < \kappa_{max}$. Field broadening parameter:

$$\mu_H \equiv \frac{\Delta\omega_R / \omega_n}{\Delta\omega / \omega_{n0}} = Nn\left|\frac{\Delta\omega_R}{\omega_n}\right|,$$

$$\mu_H\Big|_{\kappa_1=\kappa_{\max}} \approx 1,$$

for distinguished line shape

Broadening by Off the Axis Effects

- Half-width of the broadening due to emittance:

$$\left|\left\langle \frac{\Delta\omega}{\omega} \right\rangle\right|_{x,y} = \frac{\gamma^2}{1+k^2/2}\Theta_{x,y}^2 , \qquad \Theta_{x,y} = \varepsilon_{x,y} / \sigma_{x,y}$$

- Emittance broadening parameter: $$\mu_{x,y} = nN\left|\left\langle \frac{\Delta\omega}{\omega} \right\rangle\right|_{x,y}$$

- $\Theta_{x,y}$ — electron beam angular divergences in the undulator, $\varepsilon_{x,y}$— horizontal and the vertical emittances of the beam, $\sigma_{x,y}$ — the beam size.

- Is the broadening dominated by the angular divergence of the beam or by the constant magnetic field? It depends on which bending angle is bigger: $\Theta_{x,y} > \theta_H$ or $\Theta_{x,y} < \theta_H$

Combined — Field and Off the Axis Effect on the UR Spectrum

- UR spectrum with off the axis and with field effect:

$$\omega_{\mathrm{n}}\Big|_{\psi\neq 0,\ B_d\neq 0} = n\omega_R = \frac{2n\omega_0\gamma^2}{\left(1+\dfrac{k^2}{2}\right)+(\gamma\psi)^2+(\gamma\theta_H)^2-\sqrt{3}(\gamma\theta_H)(\gamma\Omega)}$$

$$\Omega = \psi\left(\rho\sin\varphi - \kappa\cos\varphi\right)/\kappa_1\,.$$

- Constant field and off the axis effect can compensate each other, but NOT completely!

Off the Axis Effects Compensation

- Angular divergence is compensated by bending angle:

$$\tilde{\theta}_H = \frac{\sqrt{3}}{2}\Omega, \quad \Omega = \psi\,\frac{\rho\sin\varphi - \kappa\cos\varphi}{\kappa_1}, \qquad \kappa_1 = \sqrt{\rho^2 + \kappa^2}$$

- Horizontal (radial) divergence compensation by vertical field component:

$$\tilde{\theta}_H = \mp\psi\,\frac{\sqrt{3}}{2}\,\frac{\kappa}{\kappa_1}, \quad \varphi = 0, \pi.$$

- Vertical divergence compensation by horizontal field component:

$$\tilde{\theta}_H = \pm\psi\,\frac{\sqrt{3}}{2}\,\frac{\rho}{\kappa_1}, \quad \varphi = \pm\frac{\pi}{2}.$$

- Best compensation (in half-plane) up to: $\psi \rightarrow \psi / 2.$

(with very little on-axis fade!) for $|\beta| < 10$.

Broadening Due to Beam Energy Spread

- Broadening due to the electron energy spread in beam:

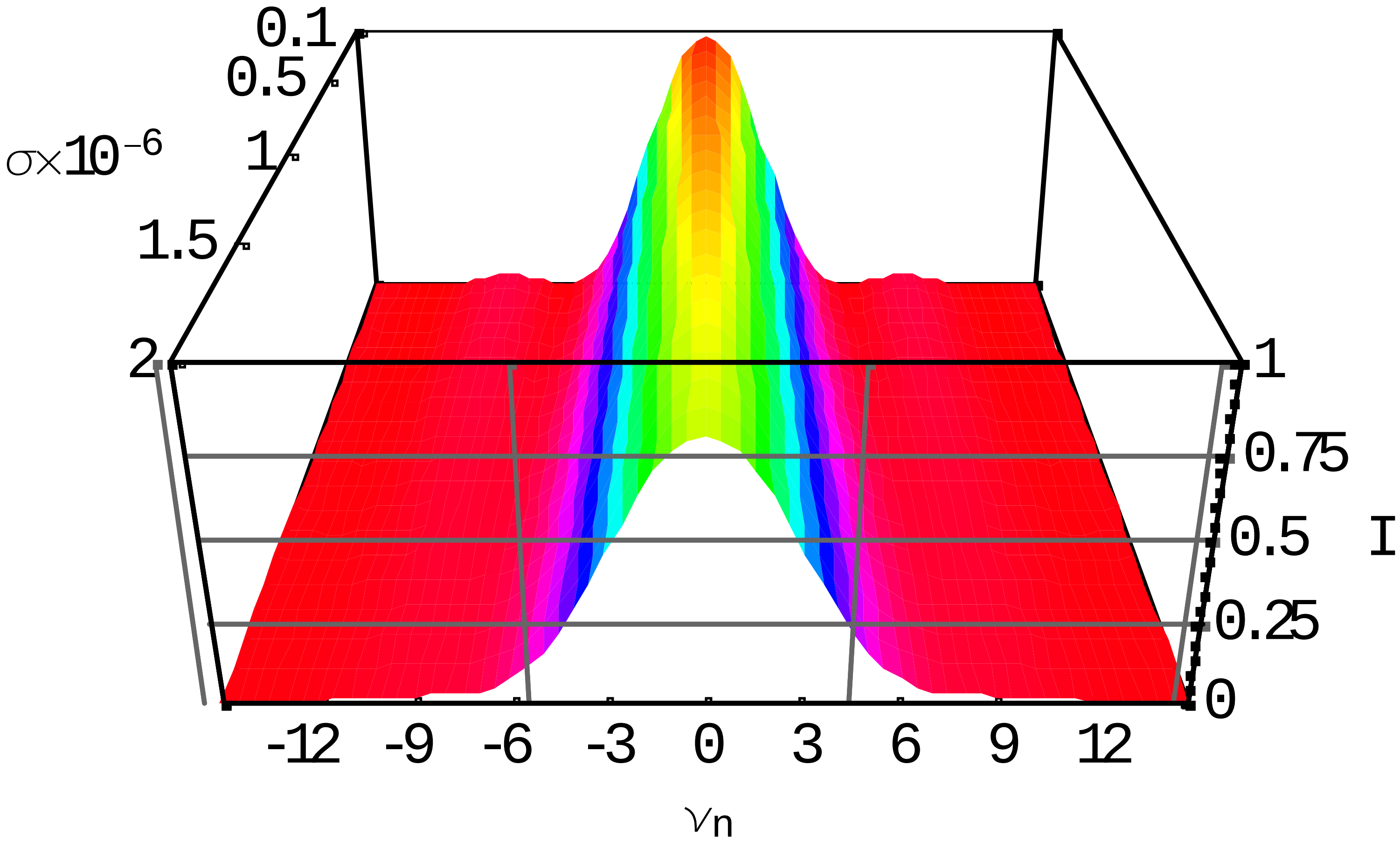
$$I = \int_{-\infty}^{\infty} \left(\left| S\left(\nu_{\mathrm{n}} + 4\pi n N\varepsilon, \beta\right) \right| \right)^2 \exp(-\varepsilon^2 / 2\sigma_e) \Big/ \sqrt{2\pi\sigma_e}\, d\varepsilon$$

- Relevant line broadening is $(\Delta\omega / \omega)_\varepsilon \approx 2\sqrt{\sigma_e}$ with zero average shift.
- Energy spread broadening parameter:

$$\mu_\varepsilon \equiv \frac{(\Delta\omega / \omega)_\varepsilon}{(\Delta\omega / \omega)_0} \approx 4Nn\sqrt{\sigma_e}$$

- Broadening effects, induced by energy spread in beams, are negligible when $\sqrt{\sigma_e} << 1/(4nN)$

Broadening Due to Beam Energy Spread



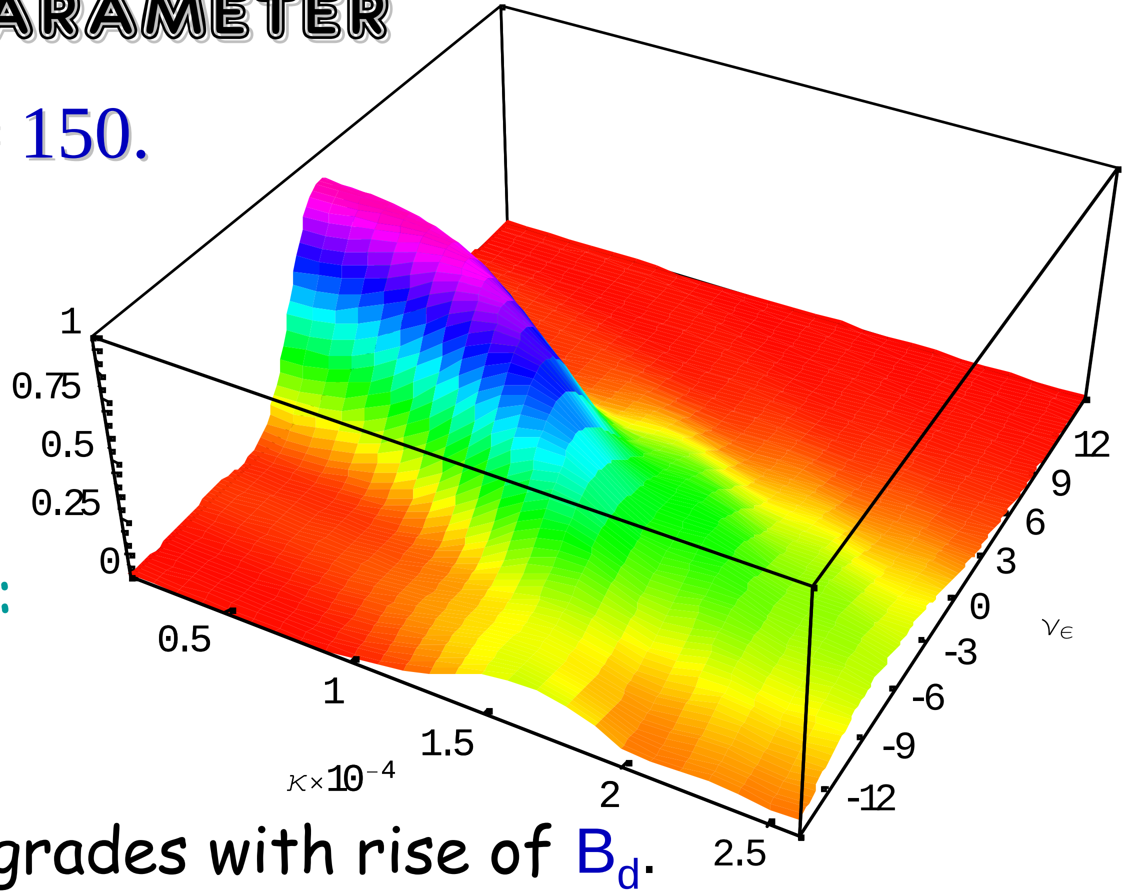
- Beam energy spread effects on the fundamental harmonic of the undulator with $k = 2$, $N = 100$.

On-axis Line Broadening for Field parameter

Emission line for $N = 150$.

$n = 1, k = 2$

- Noticeable distortion already for $B_d \sim 8 \cdot 10^{-5} B_0$.
- Reasonable limit for constant magnetic field: $B_{d\ max} \sim 10^{-4} B_0$.



- Spectral line rapidly degrades with rise of B_d.
- Undulator with $N = 150$ periods, 5 kGauss on the axis in the Earth's magnetic field 0.5 Gauss - means $B_d \sim 1 \cdot 10^{-4} B_0$ notice deterioration of spectral line — at limit of effectiveness!
- The Earth creates significant distortions when N>150!

Field Compensation of the Off-Axis Line broadening

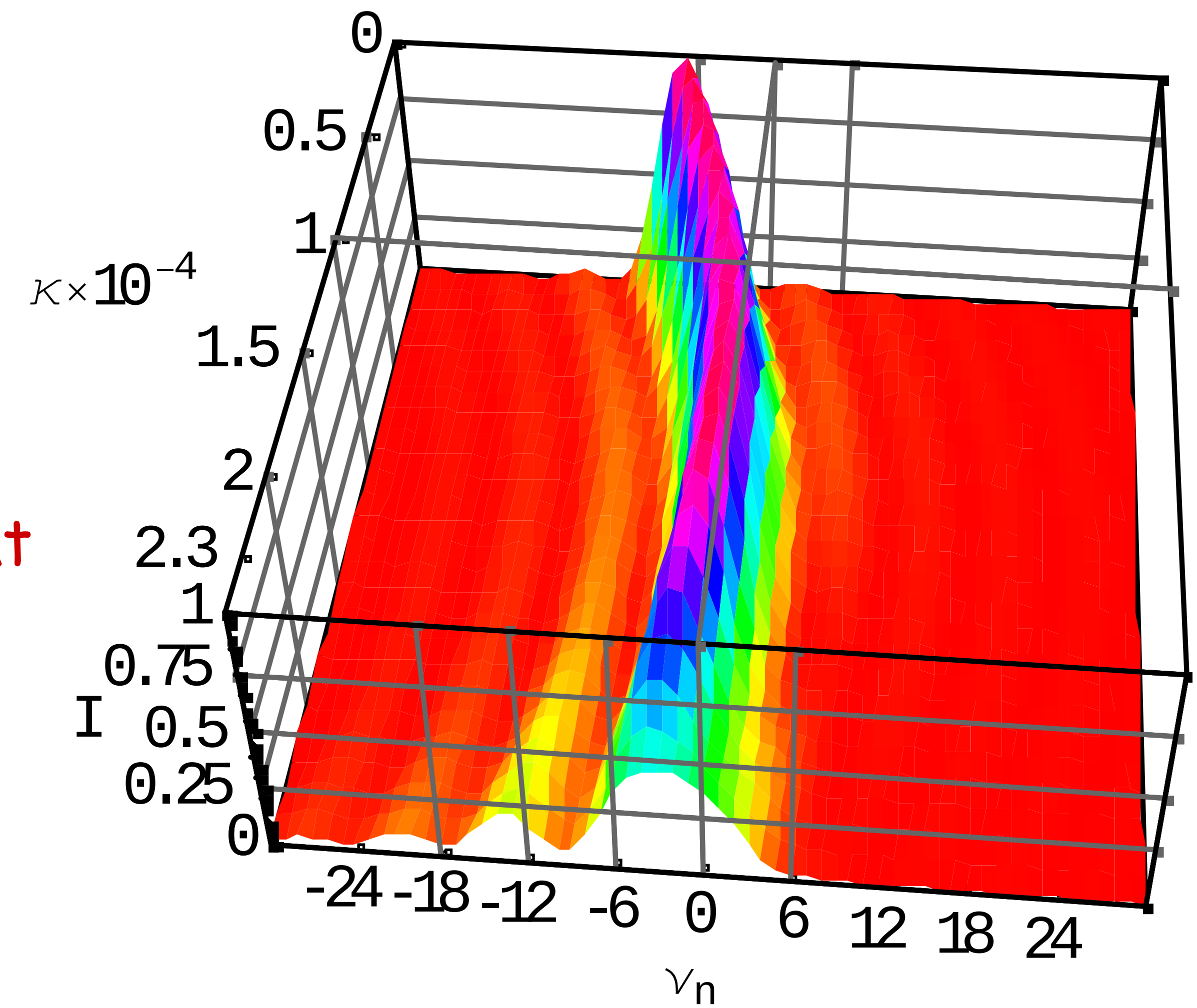
Emission line for

$N = 150$, $n = 1$, $k = 2$,

$\sigma = 10^{-10}$, off-axis angle: $\gamma\psi = 0.15$

Best compensation at $B_d \sim 1.0 \div 1.8\ 10^{-4} B_0$.

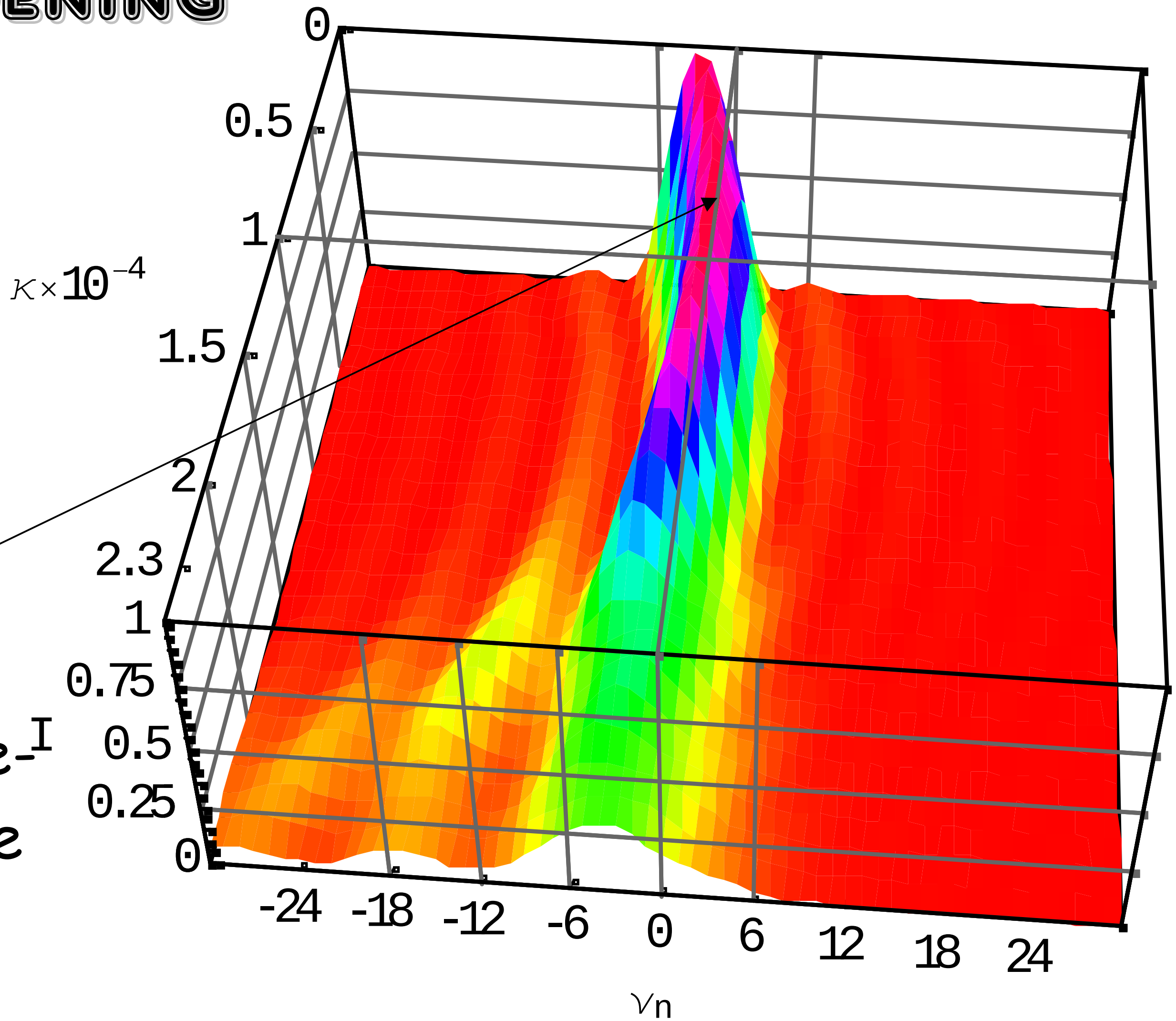
For $B_d > 2\ 10^{-4} B_0$ incoherence prevails.



- Fundamental harmonic line shift and angular divergence compensation by the constant magnetic field $B_d = \kappa_1 B_0$.

Field Compensation of Off-Axis Line broadening

Emission line for
Novosibirsk, Siberia 2:
$N = 300$, $n = 1$, $k = 0.5$, $\sigma = 10^{-8}$, $B_0 = 7.5$kG, off-axis angles: $\gamma\psi \sim 0.04$.



- Compensation at $B_d \sim 0.7\ 10^{-4} B_0 \approx 0.5$G. (for $B_d > 10^{-4} B_0$ incoherence prevails on the axis, line broadens).

- Fundamental harmonic line shift and angular divergence compensation by the constant magnetic field $B_d = \kappa_1 B_0$.

Conclusions

- Constant magnetic field can cause serious change in UR spectrum, intensity and cause line broadening in real devices.
- Underlying physics of disturbance — deviation of an electron from its regular oscillatory trajectory and disruption of the coherence of electron oscillations in undulators due to constant component of magnetic field.
- Perturbations of the electron trajectory due to the field are accumulated along the undulator length and are comparable with those due to off-axis effects, energy spread in beams and other broadening contributions.
- Constant field can partially compensate off-axis effect.
- Developed analytical method can be applied to virtually any undulator with distortions of periodic magnetic field.

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