Exponential Form of the Mixing Matrix in the Lepton Sector of the Standard Model

K.Zhukovsky
Faculty of Physics, M.V.Lomonosov Moscow State University;
G.Dattoli
ENEA, Frascati, Rome, Italy

Main topics

- Introduction: the concept of the Flavour Mixing in the Standard Model.
- Neutrino Mixing in the Standard Model - PMNS matrix. Similarities and differences with quarks
- Exponential parameterisation of the Neutrino Mixing matrix.
- Mixing and Space Rotations.
- Quark - Lepton Complementarity.
- Exponential parameterisation of the CP violating terms in the neutrino mixing matrix.
- Discussion and Conclusions.

Fermions in the Standard Model

- ✓ Each **generation** of particles exhibit **similar physical behaviour**.
- ✓ **Quarks carry colour charge** besides **weak isospin**, and they **interact with each other** via the **strong force**.
- ✓ **Leptons do not carry colour charge**, they have **weak isospin.**
- ✓ **Neutrinos have isospin +½, do not carry electric charge**, so their motion is directly influenced **only** the **weak force**.
- ✓ **Electron**, **muon** and the **tau** lepton **have isospin -½, carry** an **electric charge,** so they **interact electromagnetically**.

	Generation 1		Generation 2		Generation 3	
Quarks B=1/3	Up +1/2s; +2/3e	**u**	Charm +1/2s	**c**	Top +1/2s	**t**
	Down -1/2s;-1/3e	**d**	Strange -1/2s	**s**	Bottom -1/2s	**b**
Leptons L=1	Electron Neutrino +1/2s	ν_e	Muon Neutrino +1/2s	ν_μ	Tau Neutrino +1/2s	ν_τ
	Electron -1/2s	**e**	Muon -1/2s	μ	Tau -1/2s	τ

Flavour Mixing in the Standard Model

The concept: weak and mass eigenstates differ!

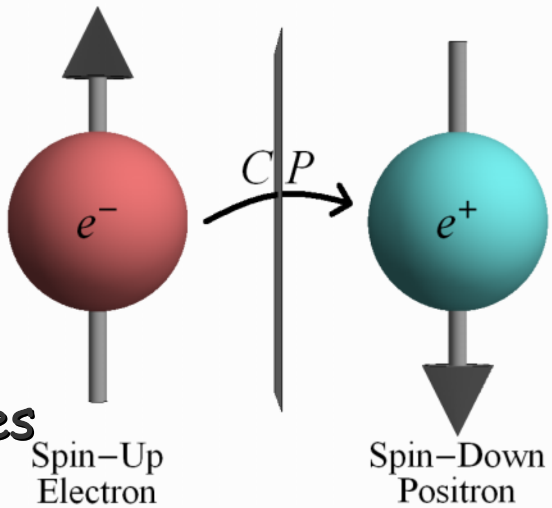
- ✓ A **quark of a given flavour is an eigenstate of the weak interaction** part of the Hamiltonian: it will interact in a definite way with the W+, W– and Z bosons.
- ✓ A **fermion of a fixed mass is an eigenstate of the kinetic and strong interaction** parts of the Hamiltonian. This mass state is a superposition of various flavours.
- ✓ The **flavour content of a quantum state may change as it propagates freely**. Thus a neutrino or quark created with one flavour can be measured with another flavour.
- ✓ **For quarks,** mixing, i.e. the **transformation from flavour to mass basis** is given by the so-called **Cabibbo-Kobayashi-Maskawa matrix** (CKM matrix).
- ✓ **For neutrinos,** the **mixing is specified by** the **PMNS matrix**. It defines the strength of flavour changes under weak interactions of neutrinos.

Charge-Parity CP in the Standard Model

- ✓ The **matrices** specify the **mismatch of quantum states** of **quarks** when they **propagate freely and** when they take **part in the weak interactions. Kobayashi, Maskawa** got **Nobel prize** for it in **2008**. (Nicola Cabibbo not awarded…)
- ✓ Both **CKM and PMNS** matrices **allow for CP violation** if there are **at least three generations**. Illustration of CP:

CP is the **product of two symmetries**:

C - for **charge conjugation**, it **transforms a particle into** its **antiparticle**.
P - for **parity**, it **creates the mirror image of a physical system**.



Neutrinos, Mixing, Notes …

- **Our understanding of neutrino physics is vaguer than that of quarks:**
- Possible existence of 4th type – sterile neutrino. ???...
- Possibly, neutrino and antineutrino are the same particle, the hypothesis first proposed by Ettore Majorana (Italy).
- The neutrino mixing was originally invented by Pontecorvo in 1957 and developed by him in 1967. One year later the solar neutrino deficit was first observed.
- **Differently from quarks, neutrino mass term** is likely to have **another origin** than that of quarks (not due to Higgs) and it is implemented through the **Majorana mass term**.
- **Flavour mixing** in the **lepton sector** can be formalised in the way, **similar** to the mixing in the **quark sector**:
- **Flavour states ν_e, ν_μ or ν_τ, are linear combinations of neutrino states with different masses, analogous to that of the bottom components of the quark pairs.**
- For 4 neutrino generations mixing mechanism is the same.

Pontecorvo-Maki-Nakagawa-Sakata (PMNS) Matrix

- **U_{PMNS} - the unitary leptonic mixing matrix - PMNS matrix :**

$$\left|\nu_\alpha\right\rangle = \sum_{i=1,2,3} U^*_{PMNS\,\alpha\, i} \left|\nu_i\right\rangle \qquad U_{PMNS\,\alpha i} \equiv \left\langle \nu_\alpha \middle| \nu_i \right\rangle$$

$U_{\alpha i}$ - the amplitude of the decay of the **W^+** boson into the lepton of the type α and the neutrino of the type i.

- Lepton mixing means that the charged **$W^\pm$ boson can couple any charged lepton mass eigenstate and any neutrino mass eigenstate** (analogy with the quark mixing).
- **Associative production** of the lepton of the type α and the neutrino state of the type α implies the **superposition of all neutrino mass eigenstates.**

U_{PMNS} Neutrino Mixing Matrix

- ✓ For 2 lepton pairs mixing is expressed via the real unitary matrix - rotation matrix in the angle θ in 2 dimensions.
- ✓ For standard 3 neutrinos theory the neutrino mixing is expressed via 3×3 unitary matrix U_{PMNS}:

$$\boldsymbol{U}_{PMNS} = \boldsymbol{U}\boldsymbol{P}_{Mjr} \qquad \boldsymbol{P}_{Mjr} = diag\left(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1\right)$$

$$\boldsymbol{V}_{Cab} = \begin{pmatrix} \cos\theta_{Cab} & \sin\theta_{Cab} \\ -\sin\theta_{Cab} & \cos\theta_{Cab} \end{pmatrix},$$

$$\boldsymbol{U} = \begin{array}{c} \\ \nu_\alpha \\ \nu_\mu \\ \nu_\tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \end{array}$$

$$s_{ij} = \sin\theta_{ij} \qquad c_{ij} = \cos\theta_{ij}$$

The mixing is given by three mixing angles θ_{12}, θ_{23}, θ_{31}, and the CP violating phases δ, α_i and α_2.

Values for Neutrino Mixing Parameters

- The phases α_1 and α_2 are non-zero only if neutrinos are Majorana particles, which effectively means that they are identical to their antiparticles. α_1 and α_2 do not influence the neutrino oscillations regardless of whether neutrinos are Majorana particles or not.
- **Matrix U** in the neutrino parameterisation matrix U_{PMNS} is **identical to the CKM matrix for quark mixing**.
- The mixing angles θ_{12} and θ_{23} are quite well determined experimentally:

Large values!

$$\theta_{12} \cong 33.9 \pm 2.4^\circ \qquad \theta_{23} \cong 45 \pm 7^\circ$$

- The third mixing angle θ_{13} is quite small, moreover, no strict experimental bound on its value! Approximate:

Small!

$$\theta_{13} \le 13^\circ$$

Possibly Zero…

Values for Neutrino Mixing Parameters

- ✓ **Neutrino mixing** is characterised by 2 **large angles** and only one small angle. **No small parameter for series expansion**! **Striking contrast with quark mixing**:
- ✓ In CKM **quark mixing** matrix **all three angles are small.** Approximate parameterisations are based on the expansion into power series of the parameter $\lambda = \sin\theta_{Cabibbo} \approx 0.22$. **EXPERIMENTAL**: The **tri-bimaximal** (TBM) **form** of the mixing matrix is consistent with the experiments :

$$\theta_{12} = \arctan(1/\sqrt{2})$$

$$\theta_{23} = \pi/4$$

$$\theta_{13} = 0$$

$$\boldsymbol{U}_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

- ✓ But there are **no physical reasons for it to be exact**! Thus, **approximations with 3 parameters for (TBM)** form appear.

Exponential Parameterisation of the PMNS Matrix

- ✓ **Exponential form of the neutrino mixing matrix.** (Dattoli, Zhukovsky)

$$\boldsymbol{U} = e^{\mathbb{A}}, \quad \mathbb{A} = \begin{pmatrix} 0 & \lambda_1 & \lambda_3 e^{i\delta} \\ -\lambda_1 & 0 & -\lambda_2 \\ -\lambda_3 e^{-i\delta} & \lambda_2 & 0 \end{pmatrix}$$

$$\cancel{?\ \lambda_3 \propto \lambda^3\ ?\ \lambda_2 \propto \lambda^2\ ?}$$

NO!

(***U***- unitary. δ accounts for the violation of CP, $\lambda_{1,2,3}$ – for the neutrino mixing).

Mixing angles θ_{12} and θ_{23} are of the order of 1, no hierarchy as with quarks!

- ✓ Notice: if $\delta=0$ i.e. for conserved CP, mixing matrix in exponential parameterisation is the angle-axis presentation of rotations in classical mechanics.

Does it hold the same as for quarks?

Classical Rotation and Exponential Parameterisation of Quark Mixing Matrix

- ✓ Exponential form of quark mixing matrix: $\hat{\boldsymbol{V}} = e^{\mathbb{A}}$, $\mathbb{A} = \begin{pmatrix} 0 & \lambda & \alpha\lambda^3 e^{i\delta} \\ -\lambda & 0 & -\beta\lambda^2 \\ -\alpha\lambda^3 e^{-i\delta} & \beta\lambda^2 & 0 \end{pmatrix}$

  (Dattoli, Zhukovsky, Eur.Phys.J.C)

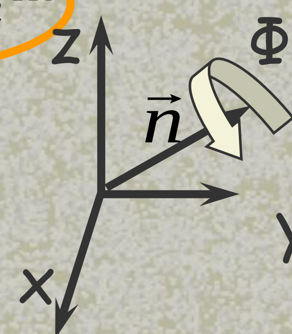
(δ accounts for the violation of CP, λ – for the quark mixing and $\alpha,\beta\sim1$)

Classical rotation matrix M: Single angle of rotation Φ and the direction unit vector

$$\boldsymbol{M} = \boldsymbol{P}_{\boldsymbol{rot}}\left(\hat{\vec{\boldsymbol{n}}}, \Phi\right) = e^{\Phi \boldsymbol{N}}$$

$$\hat{\vec{\boldsymbol{n}}} = \left(n_x, n_y, n_z\right)$$

$$\boldsymbol{N} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$



- ✓ Mixing matrix in exponential parameterisation with conserved CP (i.e. $\delta=0$) is the angle-axis presentation of rotations in classical mechanics.

Exponential Form of the MIXING MATRIX

New Parameterisation of PMNS matrix

- ✓ Let us write a new exactly unitary exponential parameterisation $\tilde{V}$ for the mixing of neutrinos (Dattoli, Zhukovsky, Eur.Phys.J.C):

$$\boxed{\tilde{V} = \boldsymbol{P}_{Rot}\boldsymbol{P}_{CP}\boldsymbol{P}_{Mjr}}$$

$$\boldsymbol{P}_{Mjr} = e^{\boldsymbol{A}_{Mjr}} \qquad \boldsymbol{A}_{Mjr} = i\begin{pmatrix} \alpha_1/2 & 0 & 0 \\ 0 & \alpha_2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{P}_{CP} = e^{\boldsymbol{A}_{CP}} \qquad \boldsymbol{A}_{CP} = \begin{pmatrix} 0 & 0 & \mu\left(-1+e^{i\delta}\right) \\ 0 & 0 & 0 \\ \mu\left(1-e^{-i\delta}\right) & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{P}_{Rot} = e^{\boldsymbol{A}_{Rot}} = \exp\begin{pmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & -\nu \\ -\mu & \nu & 0 \end{pmatrix}$$

Notes:

- ✓ the **rotation angles in the matrix** P_{Rot} **are not the same as in** the PMNS matrix in the standard parameterisation U. ***The same only for small CP violation.***
- ✓ **No small parameter** λ in **Neutrino mixing matrix** U contrary to CKM matrix for quarks. The values of the **phase** δ as well of the **phases** α_1 and α_2 so far **remain unconstrained** by experiments.

New Parameterisation Explicitly

The parameterisation $\tilde{V}$, omitting Majorana part becomes:

$$\tilde{U} = MP_{CP} = P_{Rot}P_{CP}$$

$$\tilde{U} = \left(\Xi_1 \cos 2\Delta + \kappa^- \Xi_3 \sin 2\Delta, \quad \Xi_2, \quad \Xi_3 \cos 2\Delta + \kappa^+ \Xi_1 \sin 2\Delta\right)$$

$$\Delta = \mu \sin\frac{\delta}{2} \qquad \Xi_1 = \begin{pmatrix} M_x \\ M_{yx} \\ M_{zx} \end{pmatrix} \quad \Xi_2 = \begin{pmatrix} M_{xy} \\ M_y \\ M_{zy} \end{pmatrix} \quad \Xi_3 = \begin{pmatrix} M_{xz} \\ M_{yz} \\ M_z \end{pmatrix} \qquad \kappa^{\pm} = ie^{\pm i\frac{\delta}{2}}$$

$$M_{ij} = (1 - \cos\Phi)n_i n_j + \delta_{ij}\cos\Phi - \varepsilon_{ijk} n_k \sin\Phi$$

- ✓ Compare these expressions with the standard form ***U*** of the PMNS and express matrix rotation angle ***Φ*** and the rotation vector ***n*** in terms of the parameters of the standard form of the PMNS mixing matrix c_{ij} and s_{ij}. Relations appear cumbersome …

Analogy with the Rotations

- Compare the common generator $\boldsymbol{M}$ of 3D rotations with the rotational part $\boldsymbol{P}_{Rot}$ of the exponential parameterisation of the PMNS matrix :

$$\boldsymbol{M}\left(\hat{\vec{\boldsymbol{n}}}, \Phi\right) = e^{\Phi \boldsymbol{N}}$$

$$\hat{\vec{\boldsymbol{n}}} = \left(n_x, n_y, n_z\right)$$

$$\boldsymbol{N} = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$

- $\boldsymbol{M}$ – generator of rotations in 3D. It is the same as $\boldsymbol{P}_{rot}$.
- **Parameters** of the purely **rotational part** of the neutrino **mixing matrix** in exponential parameterisation related **to the angle and the axis of the three dimensional rotation**:

$$n_z = -\frac{\lambda}{\Phi} \qquad n_y = \frac{\mu}{\Phi} \qquad n_x = \frac{\nu}{\Phi} \qquad \Phi = \pm\sqrt{\lambda^2 + \mu^2 + \nu^2}$$

The Entries of the Exponential Matrix Parameterisation of the PMNS

- ✓ **Compare the exponential form with TBM matrix values, which agree with experiment, we obtain for parameters** (Dattoli, Zhukovsky, Eur. Phys. J. C)**:**

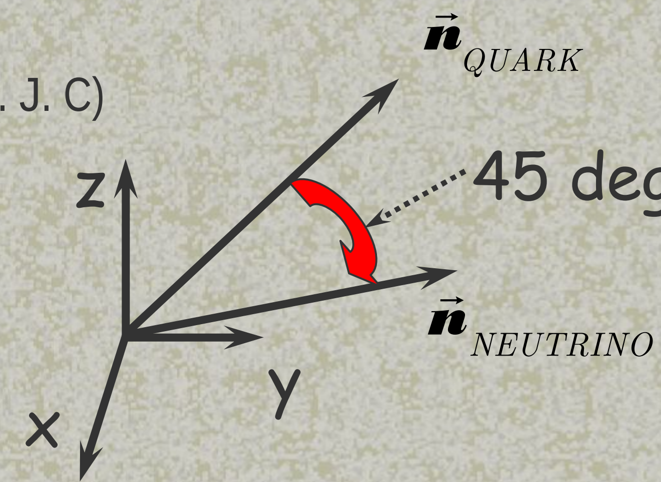
$$\lambda \cong 0.5831,\ \mu \cong -0.2415,\ \nu \cong -0.7599$$

- ✓ **$|\mu|$ is rather small,** compared with *1*, and hence we could possibly make use of it, performing the expansion in series of this small parameter:

$$\boldsymbol{U} \cong \boldsymbol{P}_{\boldsymbol{Rot}}\boldsymbol{P}_{\boldsymbol{CP}},\ \mu\delta << 1$$

Quark –Lepton Complementarity

- Calculate the **coordinates of the rotation vector *n* for the exponential form of the PMNS** neutrino mixing matrix **and CKM** quark mixing matrix **from the experimental data.**
- Surprise! The value of the **angle between these vectors is practically *45* degrees**.
  (Dattoli, Zhukovsky, Eur. Phys. J. C)



- The fact that the **rotation axes for the neutrino and the quark mixing form the angle,** very close to ***45* degrees** is another way to formulate the **hypothesis of equality and complementary angles for quarks and neutrinos**!

CP-Majorana Terms Together

- ✓ Now let's combine the CP violating term and the Majorana term in one, including parameter δ and α, with the help of the following identity:

$$e^{\boldsymbol{A}_{CP}} e^{\boldsymbol{A}_{Mjr}} \cong e^{\boldsymbol{A}_{CP}+\boldsymbol{A}_{Mjr}}\left(1+\frac{1}{2}\left[\boldsymbol{A}_{CP},\boldsymbol{A}_{Mjr}\right]\right)$$

- ✓ The commutator is of the order of $O(\alpha_1\mu\delta)$.
  Suppose that the parameters α and δ are small
  Then, we obtain $\boldsymbol{V}_{MCP}$ matrix -

$$\boldsymbol{V}_{MCP} = \boldsymbol{P}_{CP}\boldsymbol{P}_{Mjr} \cong \exp\begin{pmatrix} i\frac{\alpha_1}{2} & 0 & \left(-1+e^{i\delta}\right)\mu \\ 0 & i\frac{\alpha_2}{2} & 0 \\ \left(1-e^{-i\delta}\right)\mu & 0 & 0 \end{pmatrix} + O(\alpha_1\mu\delta)$$

Parameters of CP-Majorana Matrix V_{MCP}

$$\boldsymbol{V}_{MCP} = \boldsymbol{P}_{CP}\boldsymbol{P}_{Mjr} = \begin{pmatrix} \xi_1 \cos 2\Delta & 0 & \kappa^+ \sin 2\Delta \\ 0 & \xi_2 & 0 \\ \xi_1 \kappa^- \sin 2\Delta & 0 & \cos 2\Delta \end{pmatrix}$$

$$\xi_{1,2} = e^{i\frac{\alpha_{1,2}}{2}}$$

$$\kappa^{\pm} = ie^{\pm i\frac{\delta}{2}}$$

$$\Delta = \mu \sin\frac{\delta}{2}$$

- ✓ V_{MCP} matrix – the factor in the exponential parameterisation of the PMNS matrix, responsible for the CP violation effects
- ✓ V_{MCP} is not symmetric with respect to the parameters $\xi_{1,2}$ and the Majorana term can interplay with the δ phase.
- ✓ When $\alpha_1=0$, the symmetric form of the $\boldsymbol{V}_{MCP}$ is restored.
- ✓ For $\xi_2=1$, i.e. $\alpha_2=0$, the form of the V_{MCP} matrix reminds the form of the mixing matrix for 2 lepton generations, acting on an electron, taon and correspondent neutrinos with the weights ξ_1 for the entry of the mixing matrix (1,1), κ^+ for the entry (3,1) and $\xi_1 \kappa^-$ for the entry (1,3) of the mixing matrix.

Neutrino under the action of V_{MCP}

- For non-zero value of α_2 the vector of the mixed neutrino states under the action of the V^*_{MCP} matrix writes as follows (Dattoli, Zhukovsky, Eur. Phys. J. C):

$$\left|\tilde{\nu}_\alpha\right\rangle = \begin{pmatrix} \left|\nu_1\right\rangle \xi_1^{\,*} \cos 2\Delta + \left|\nu_3\right\rangle \kappa^{+^*} \sin 2\Delta \\ \left|\nu_2\right\rangle \xi_2^{\,*} \\ \left|\nu_1\right\rangle \xi_1^{\,*} \kappa^{-^*} \sin 2\Delta + \left|\nu_3\right\rangle \cos 2\Delta \end{pmatrix}$$

- **unitarity of $\boldsymbol{V_{MCP}}$ matrix and unitarity of $\boldsymbol{V}$ (PMNS) :**

$$\boldsymbol{V}_{MCP}^{-1} \cdot \boldsymbol{V}_{MCP} = \boldsymbol{V}_{MCP}^{+} \cdot \boldsymbol{V}_{MCP} = \boldsymbol{I} \qquad \tilde{\boldsymbol{V}}^{-1} \cdot \tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}^{+} \cdot \tilde{\boldsymbol{V}} = \boldsymbol{I}$$

Complete PMNS Transform with V_{MCP}

- ✓ **The neutrino vector after the PMNS transformation:**

$$\begin{pmatrix} |\bar{\nu}_e\rangle \\ |\bar{\nu}_\mu\rangle \\ |\bar{\nu}_\tau\rangle \end{pmatrix} = \boldsymbol{M} \cdot \boldsymbol{V}^*{}_{\boldsymbol{MCP}} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix} = F|\nu_1\rangle + G|\nu_2\rangle + H|\nu_3\rangle$$

$$\Delta = \mu \sin\frac{\delta}{2}$$

$$F = \xi_1{}^*\left(\Xi_1 \cos 2\Delta + \Xi_3 \kappa^{-^*} \sin 2\Delta\right)$$

$$\xi_{1,2} = e^{i\frac{\alpha_{1,2}}{2}}$$

$$G = \xi_2{}^*\Xi_2$$

$$H = \xi_2{}^*\left(\Xi_1 \kappa^{+^*} \sin 2\Delta + \Xi_3 \cos 2\Delta\right)$$

$$\kappa^{\pm} = ie^{\pm i\frac{\delta}{2}}$$

- ✓ Thus, the contribution due to ν_2 neutrinos is affected only by α_2 Majorana phase (factor $\boldsymbol{G}$), whereas the $\boldsymbol{\nu_1}$ and $\boldsymbol{\nu_3}$ neutrino enter respectively with $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ dependent factors $\boldsymbol{F}$ and $\boldsymbol{H}$. Above transformation can be seen as a rotation in the angle 2Δ, determined by the CP phase δ with the weights Ξ and κ and their products

Discussion and Conclusions

- ✓ Analogy between mass states mixing in Standard Model with conserved CP and rotations in classical mechanics.
- ✓ Exponential mixing matrix parameterisation - rotation around a fixed axis in 3D space on angle Φ. When $\Phi = 0$, the mixing fades out since the mixing matrix becomes $\mathbf{I}$.
- ✓ CP violation breaks this symmetry - rotation axis gets complex coordinate y. Majorana phases α_i break it further.
- ✓ Rotation axes for the neutrino and the quark mixing form the angle of ≈*45* degrees with each other. Quark-Lepton complementarity. $U_{CKM} x U_{PMNS} = U_M$ direct correlation in GUT.
- ✓ Exponential parameterisation of CKM and PMNS allows generation of new unitary parameterisations with distinguished CP violating part.