Fractional Analytic Perturbation Theory

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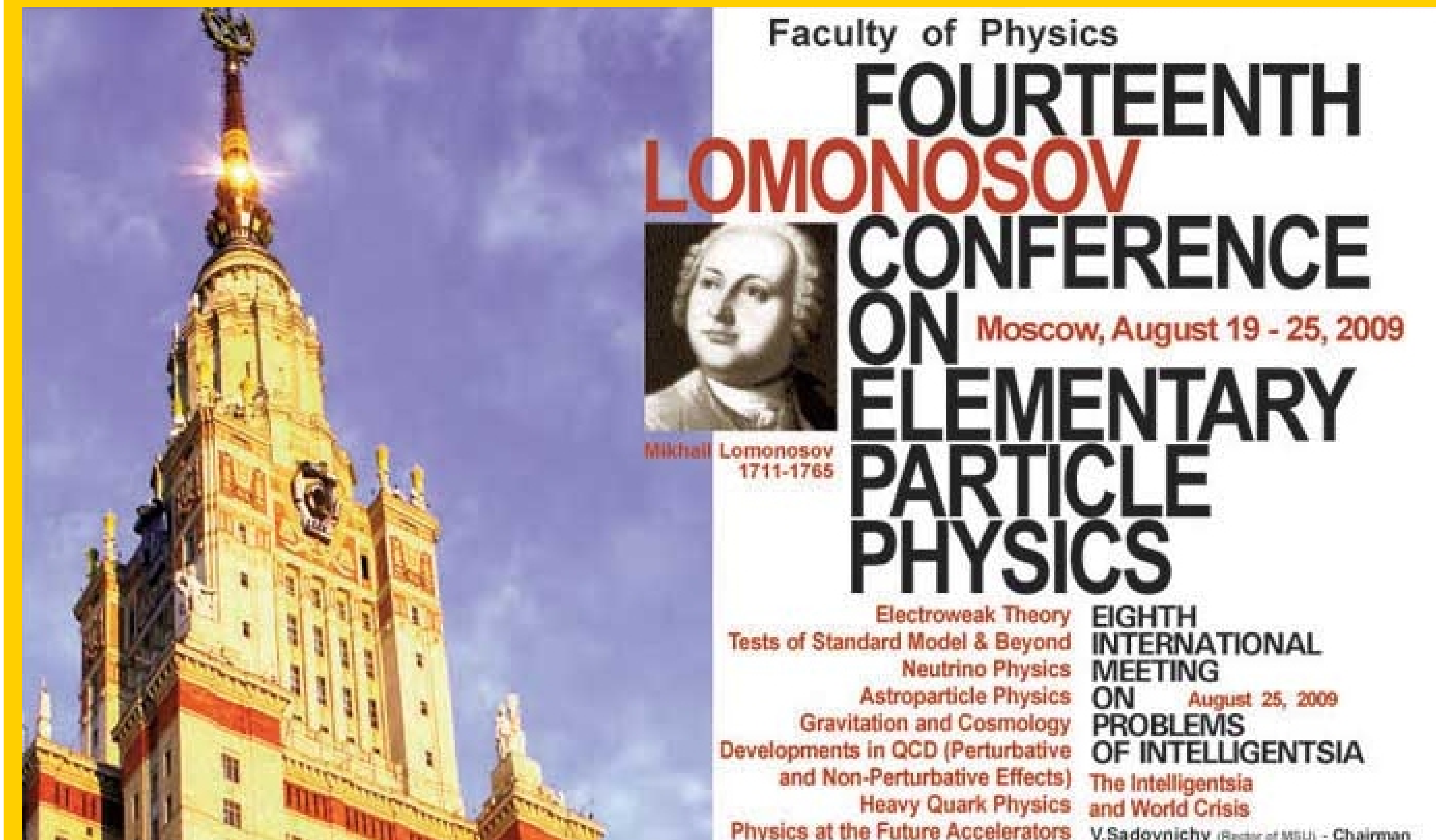

OUTLINE

- Intro: Analytic Perturbation Theory (**APT**) in QCD
- Problems of **APT** and their resolution in **FAPT**:
- Technical development of **FAPT**: thresholds
- Resummation in **APT**and **FAPT**
- Applications: Higgs decay $H^0 \to b\bar{b}$
- Conclusions

Collaborators & Publications

Collaborators:

S. Mikhailov (Dubna), N. Stefanis (Bochum), and
A. Karanikas (Athens)

Publications:

- A. B., Mikhailov, Stefanis — **PRD 72 (2005) 074014**
- A. B., Karanikas, Stefanis — **PRD 72 (2005) 074015**
- A. B., Mikhailov, Stefanis — **PRD 75 (2007) 056005**
- A. B.&Mikhailov — “Resummation in (F)APT”, **arXiv:0803.3013 [hep-ph]**
- A. B. — “Global FAPT in QCD with Selected Applications”, **Phys. Part. Nucl. 40 (2009) 715**

Analytic Perturbation Theory

in

QCD

History of APT

Euclidean	Minkowskian
$Q^2 = \vec{q}^{\,2} - q_0^2 \geq 0$	$s = q_0^2 - \vec{q}^{\,2} \geq 0$

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RG+Analyticity

ghost-free $\overline{\alpha}_{\text{QED}}(Q^2)$

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pQCD+RG: resum π^2-terms

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Arctg(s) at **LE** region

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$\mathcal{R}\left[\overline{\alpha}_s\right] \longrightarrow$ **Arctg**(s)

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Transforms: $\hat{\mathcal{D}} = \hat{\mathcal{R}}^{-1}$

Couplings: $\alpha_{\mathsf{E}}(Q^2) \Leftrightarrow \alpha_{\mathsf{M}}(s)$

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Analytic (global) pQCD$+$Analyticity

Global couplings: $\mathcal{A}_n(Q^2) \Leftrightarrow \mathfrak{A}_n(s)$

Non-Power perturbative expansions

Shirkov 1999–2001

Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s[L]$ with $L = \ln(\mu^2/\Lambda^2)$
- RG equation $\dfrac{d\, a_s[L]}{d\, L} = -a_s^2 - c_1\, a_s^3 - \ldots$
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- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$

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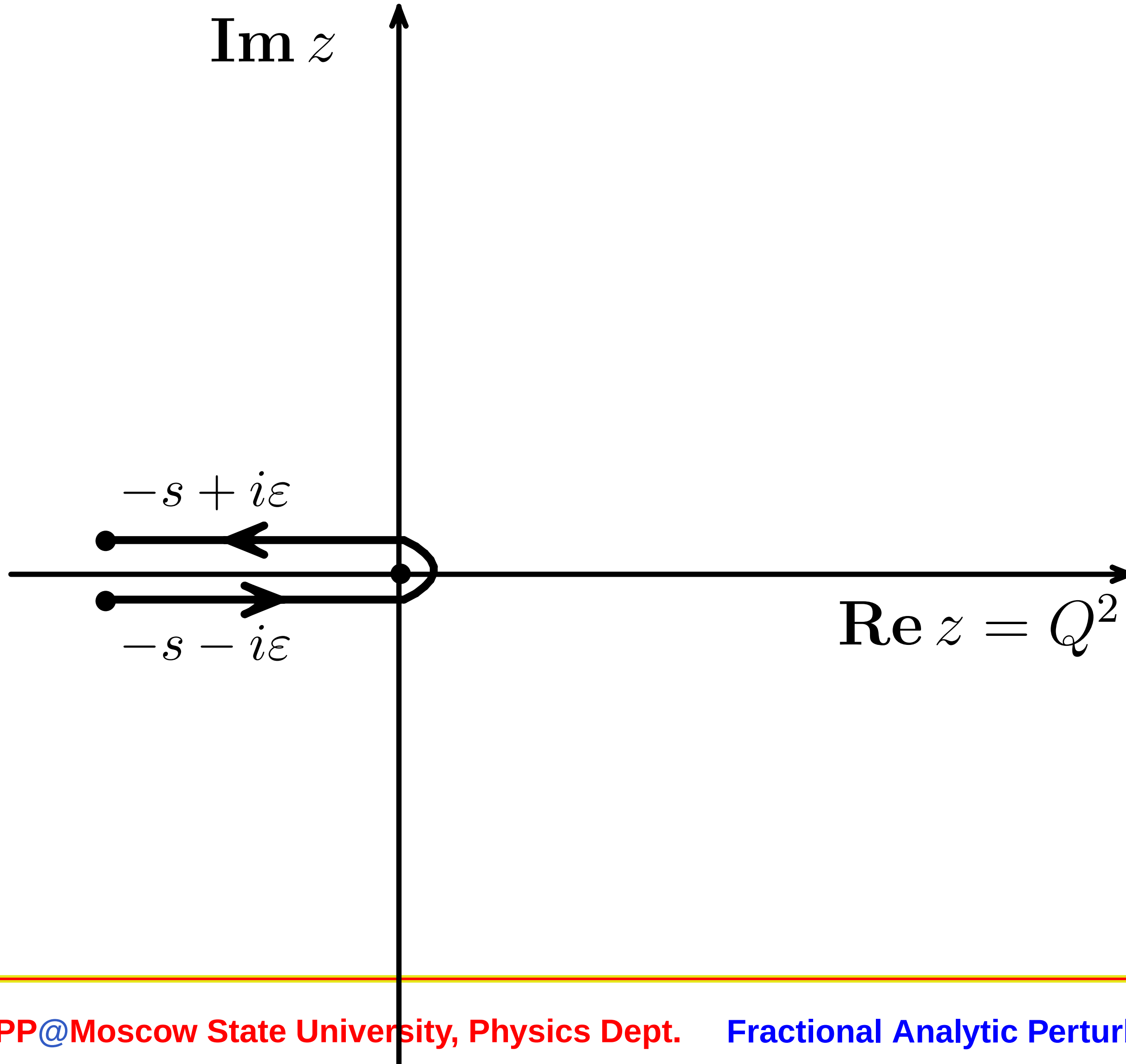
- PT series: $D[L] = 1 + d_1 a_s[L] + d_2 a_s^2[L] + \ldots$
- RG evolution: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$ reduces in 1-loop approximation to

$$Z \sim a^{\nu}[L]\Big|_{\nu\,=\,\nu_0\,\equiv\,\gamma_0/(2b_0)}$$

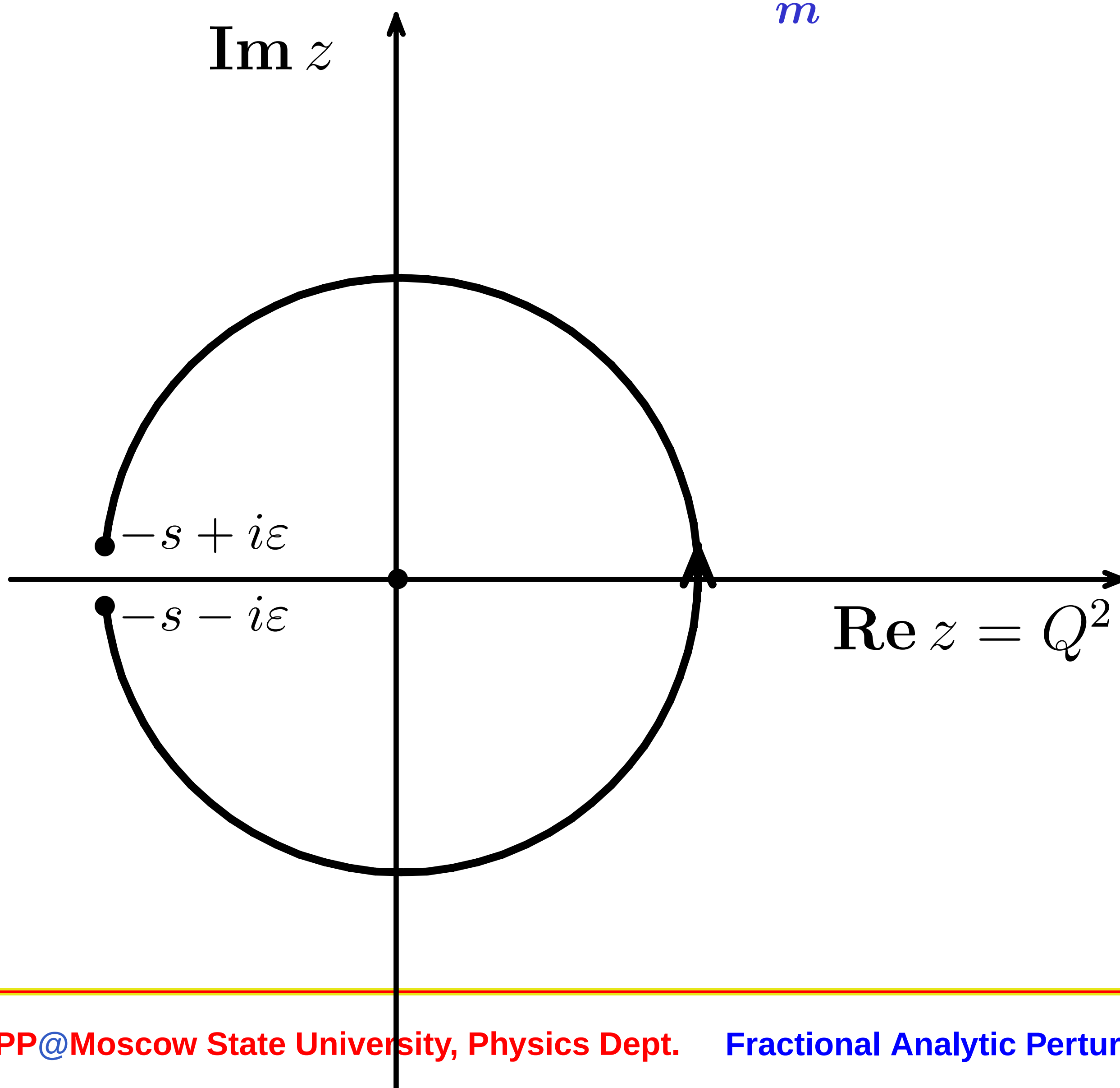
Problem in QCD PT: Minkowski region?

Quantities in Minkowski region $= \oint f(z)D(z)dz$.



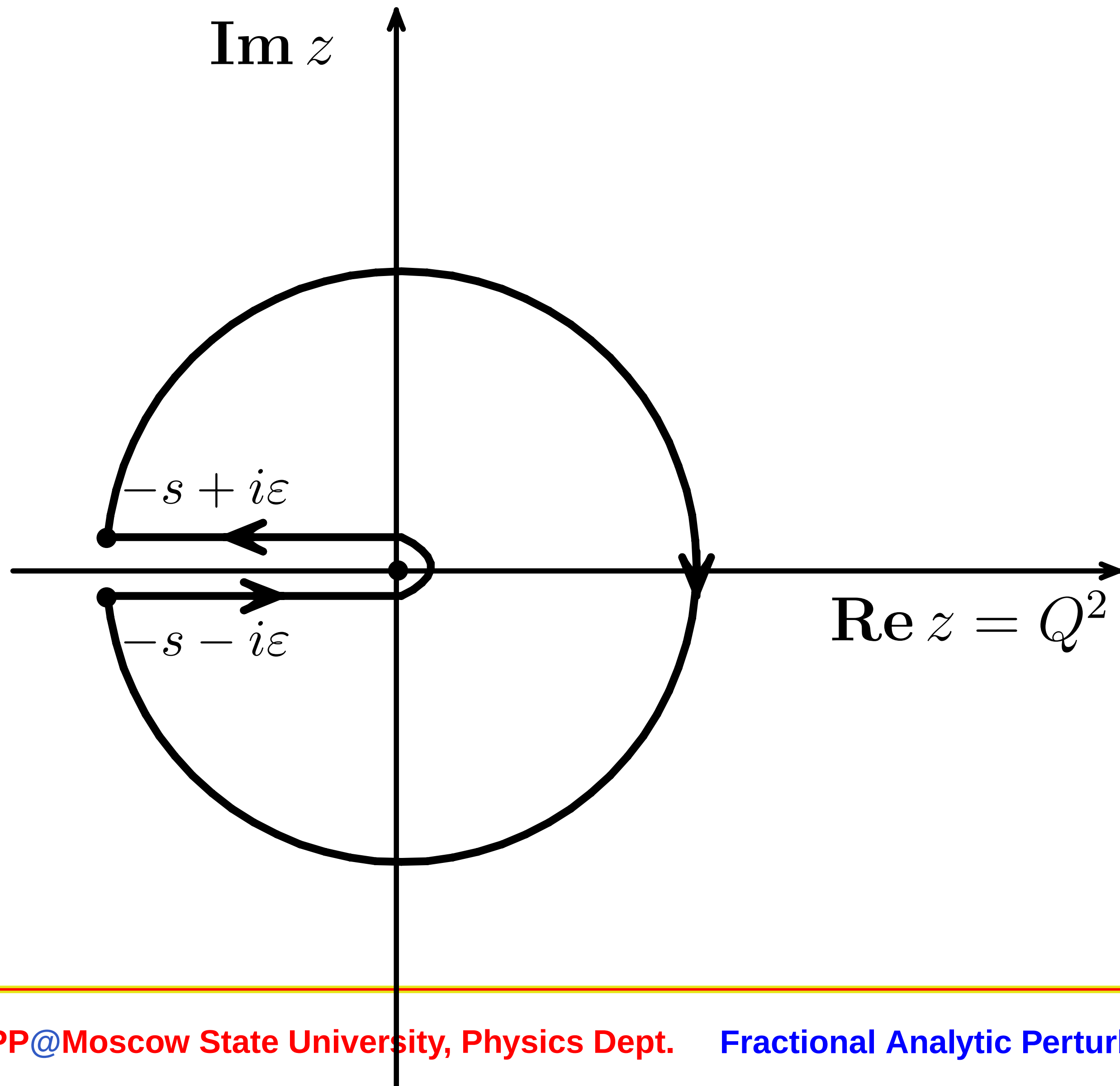
Problem in QCD PT: Minkowski region?

In $\oint f(z)D(z)dz$ one uses $D(z) = \sum_m d_m \alpha_s^m(z)$.



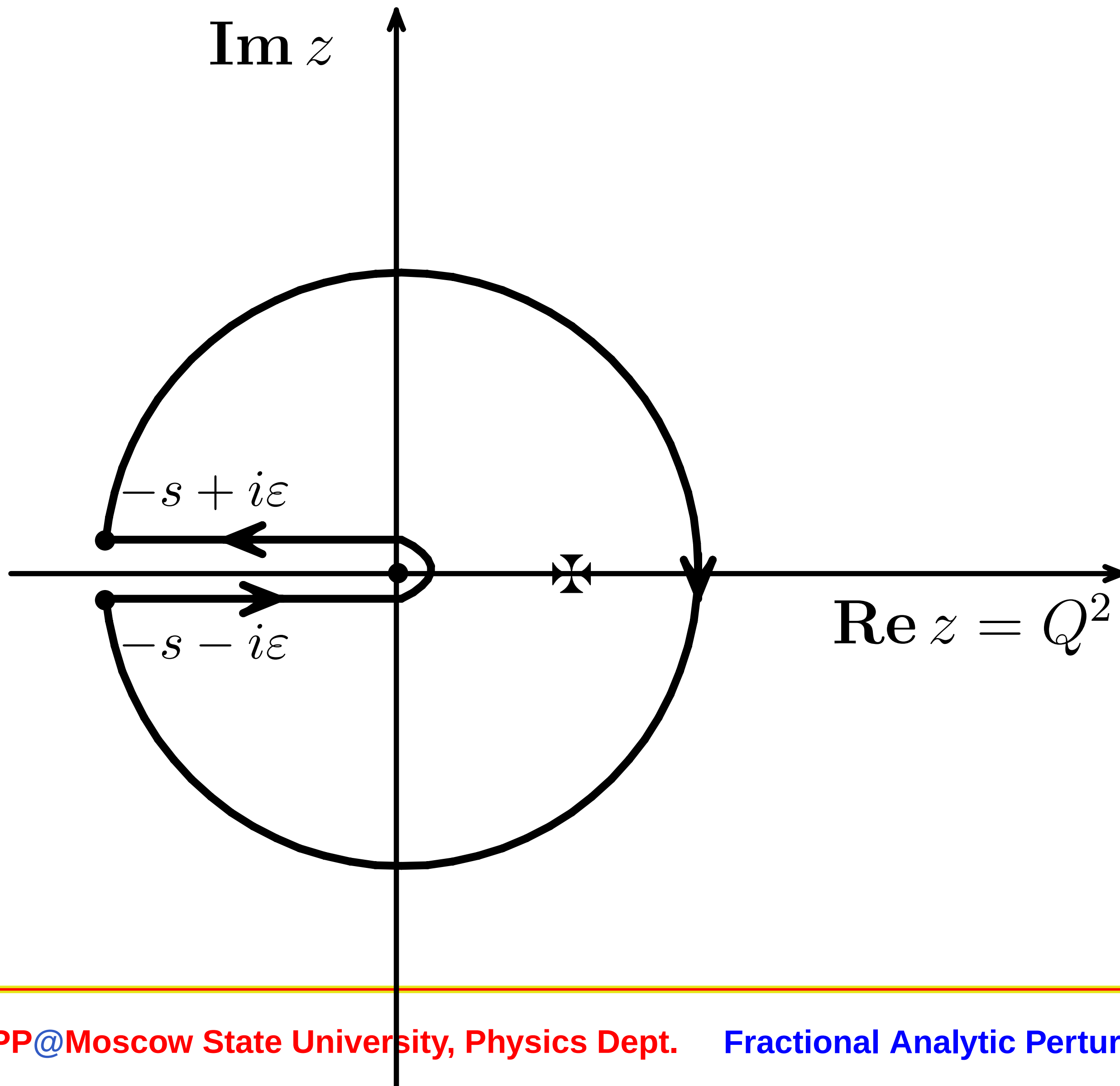
Problem in QCD PT: Minkowski region?

This change of integration contour is legitimate if $D(z)f(z)$ is analytic inside



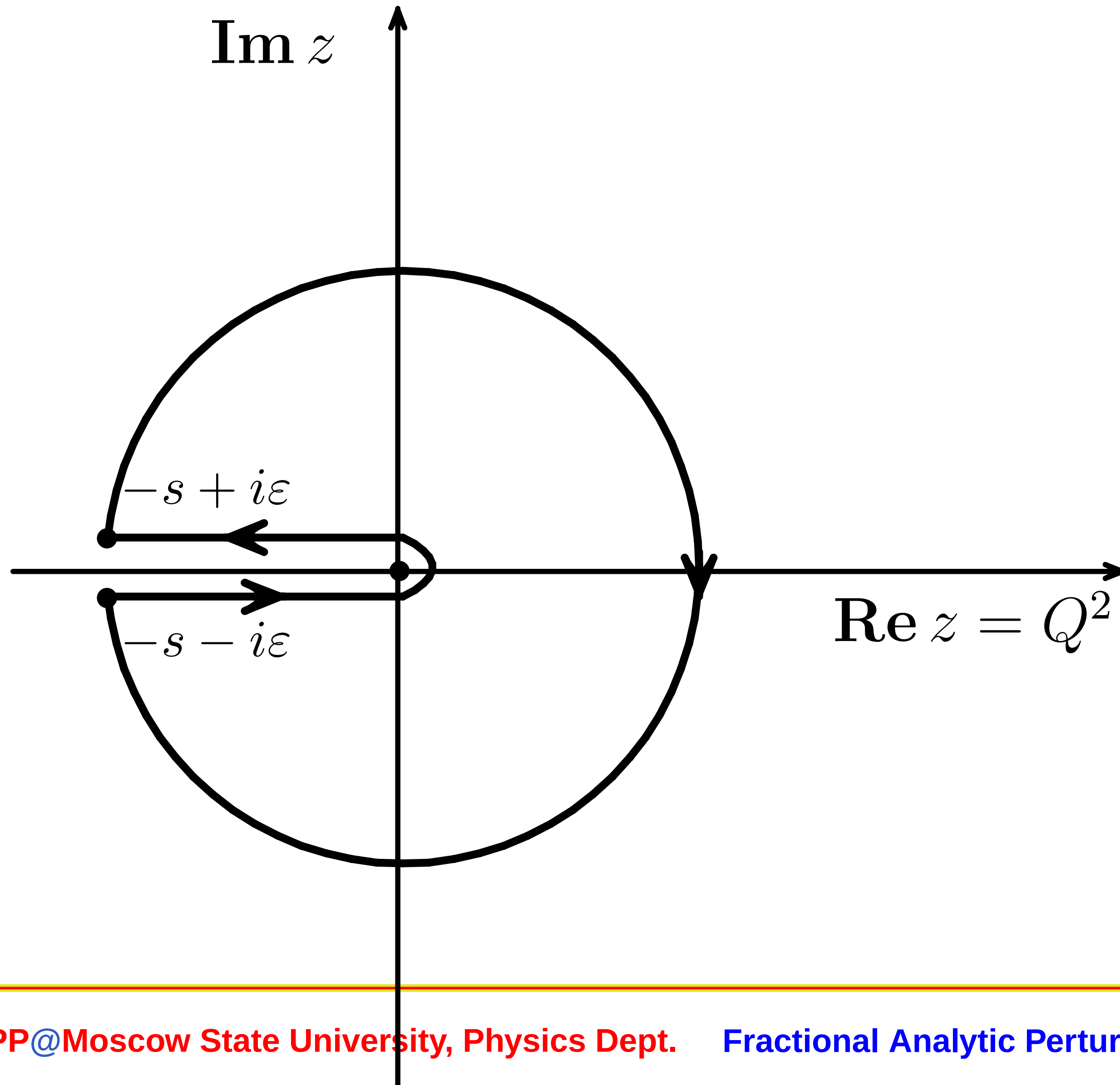
Problem in QCD PT: Minkowski region?

But $\alpha_s(z)$ and hence $D(z)f(z)$ have Landau pole singularity just inside!



Problem in QCD PT: Minkowski region?

In **APT** effective couplings $\mathcal{A}_n(z)$ are analytic functions $\Rightarrow$ Problem does not appear! Equivalence to CIPT for $\boldsymbol{R(s)}$.



*Basics of **APT***

- Different effective couplings in **Euclidean** **(S&S)** and **Minkowskian** **(R&K&P)** regions

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  $\Downarrow$ UV asymptotics $\qquad$ $\Downarrow$ Spectrality

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- Euclidean: $-q^2 = Q^2, \;\; L = \ln Q^2/\Lambda^2, \;\; \{\mathcal{A}_n(L)\}_{n\in\mathbb{N}}$
- Minkowskian: $q^2 = s, \;\; L_s = \ln s/\Lambda^2, \;\; \{\mathfrak{A}_n(L_s)\}_{n\in\mathbb{N}}$

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- $\boxed{\text{PT}}$ $\quad \sum\limits_m d_m a_s^m(Q^2) \quad \Rightarrow \quad \sum\limits_m d_m \mathcal{A}_m(Q^2)$ $\quad \boxed{\textbf{APT}}$

  m is power $\quad \Rightarrow \quad$ m is **index**

Spectral representation

By **analytization** we mean “Källen–Lehmann” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

Then (note here **pole remover**):

$$\begin{aligned}
\rho(\sigma) &= \frac{1}{L_\sigma^2 + \pi^2} \\
\mathcal{A}_1[L] &= \int_0^\infty \frac{\rho(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{L} - \frac{1}{e^L - 1} \\
\mathfrak{A}_1[L_s] &= \int_s^\infty \frac{\rho(\sigma)}{\sigma}\, d\sigma = \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}
\end{aligned}$$

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with spectral density $\rho_f(\sigma) = \mathbf{Im}\left[f(-\sigma)\right]/\pi$. Then:

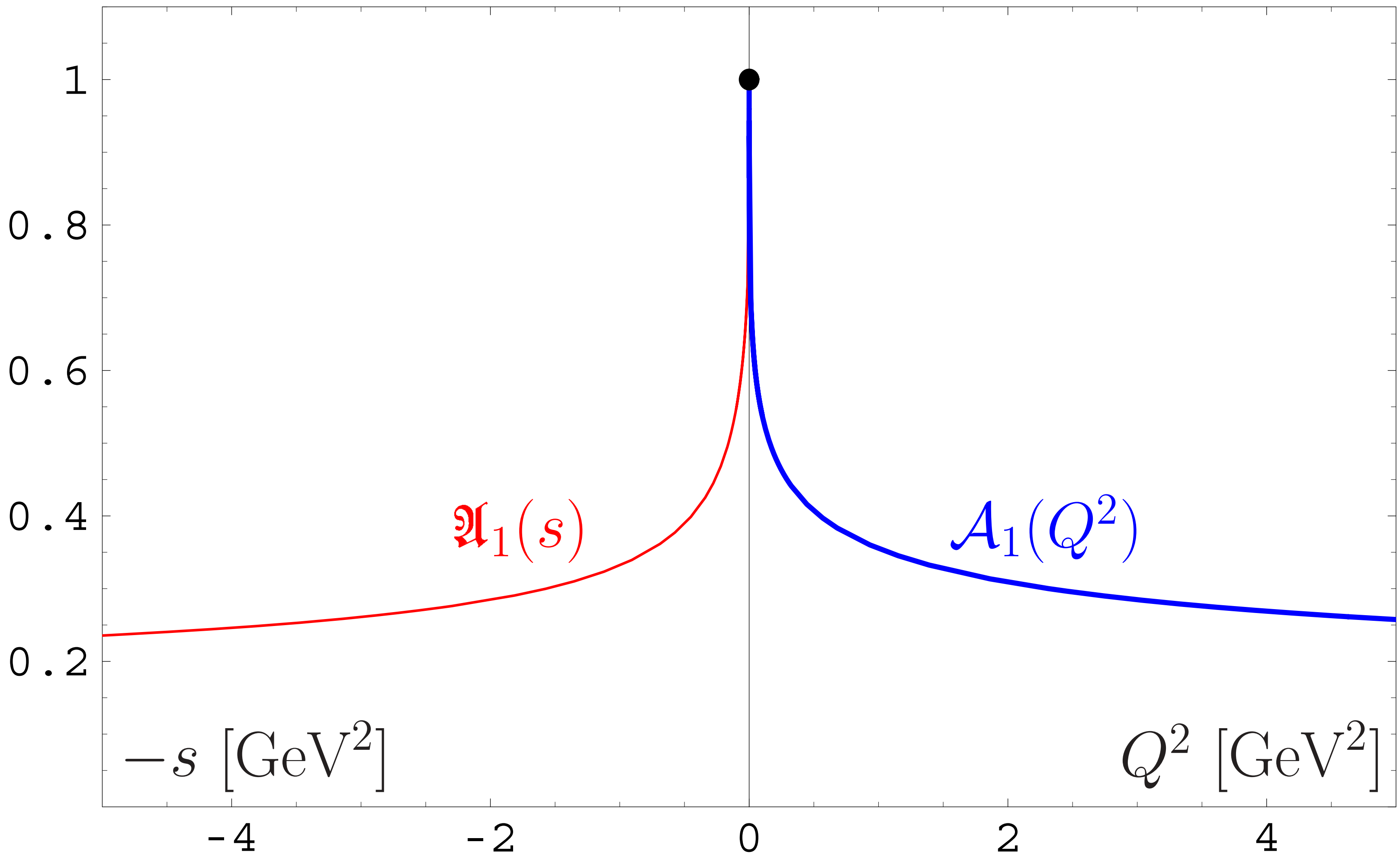
$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma = \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1} \mathfrak{A}_1[L_s]$$

$$a_s^n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s[L]$$

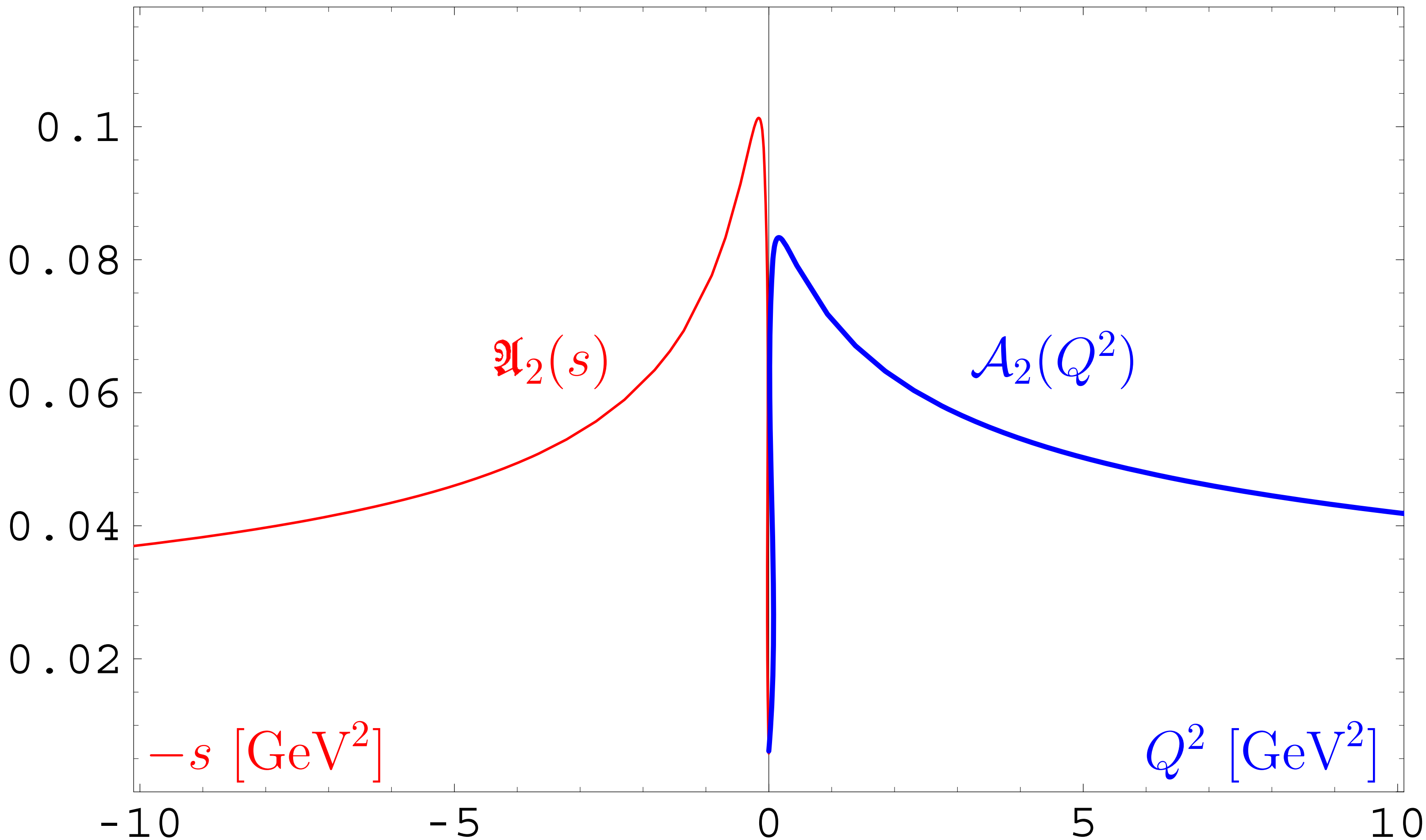
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



Problems of APT. Resolution: Fractional APT

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- RG-improvement to account for higher-orders $\longrightarrow$

$$Z[L] = \textbf{exp}\left\{\int^{a_s[L]} \frac{\gamma(a)}{\beta(a)}\, da\right\} \xrightarrow{\textbf{1-loop}} \left[a_s[L]\right]^{\gamma_0/(2\beta_0)}$$

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New functions: $(a_s)^\nu\,,\ (a_s)^\nu\, \mathbf{ln}(a_s),\ (a_s)^\nu\, L^m,\ e^{-a_s},\ \ldots$

Constructing one-loop FAPT

In one-loop **APT** we have a very nice recurrence relation

$$\mathcal{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathcal{A}_1[L]$$

and the same in Minkowski domain

$$\mathfrak{A}_n[L] = \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\mathfrak{A}_1[L]\,.$$

We can use it to construct **FAPT**.

FAPT(E): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ($L = L(Q^2)$):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced **Lerch** transcendent. function. It is analytic function in ν.

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- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_m[\pm\infty] = 0$ for $m \geq 2\,,\ m \in \mathbb{N}$;

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$):

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Here we need only elementary functions.

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Here we need only elementary functions. Properties:

- $\mathfrak{A}_0[L] = 1$;
- $\mathfrak{A}_{-1}[L] = L$;
- $\mathfrak{A}_{-2}[L] = L^2 - \dfrac{\pi^2}{3}$, $\mathfrak{A}_{-3}[L] = L\left(L^2-\pi^2\right)$, $\ldots$;
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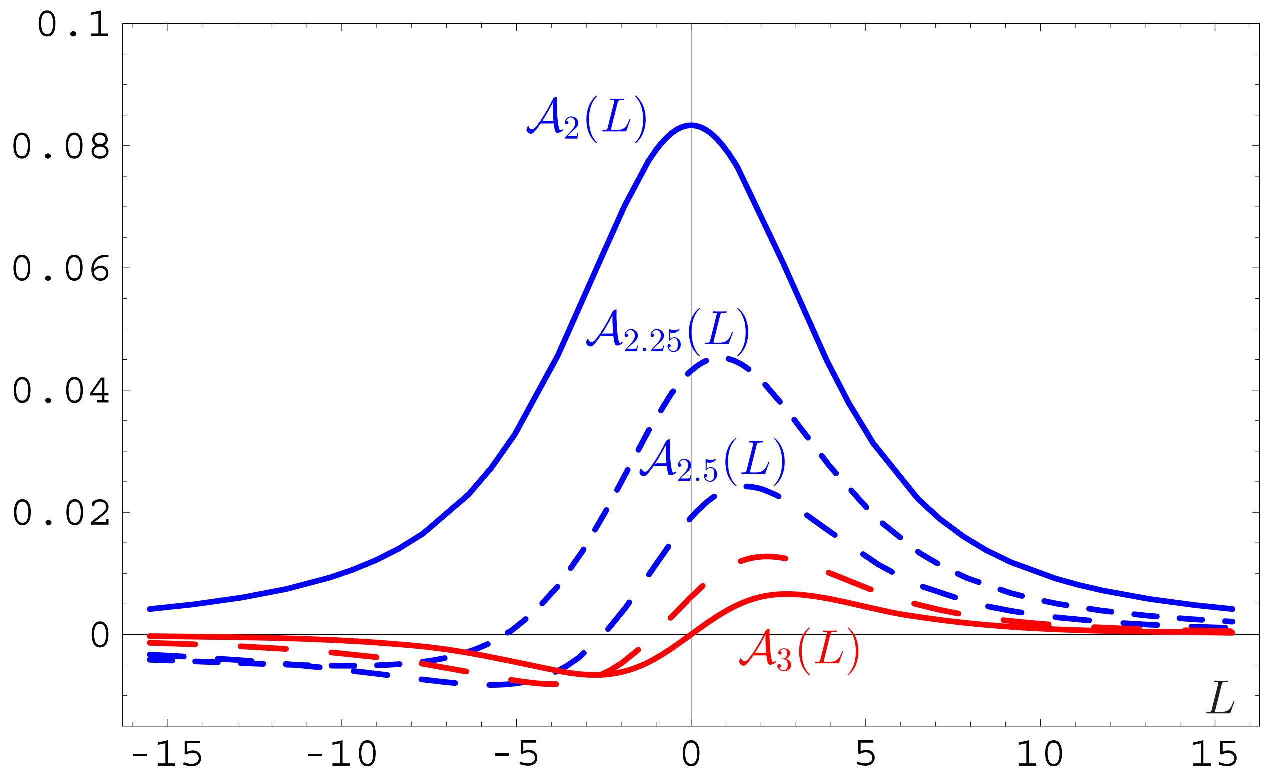
FAPT(E): Graphics of $\mathcal{A}_\nu[L]$ vs. L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Graphics for fractional $\nu \in [2, 3]$:



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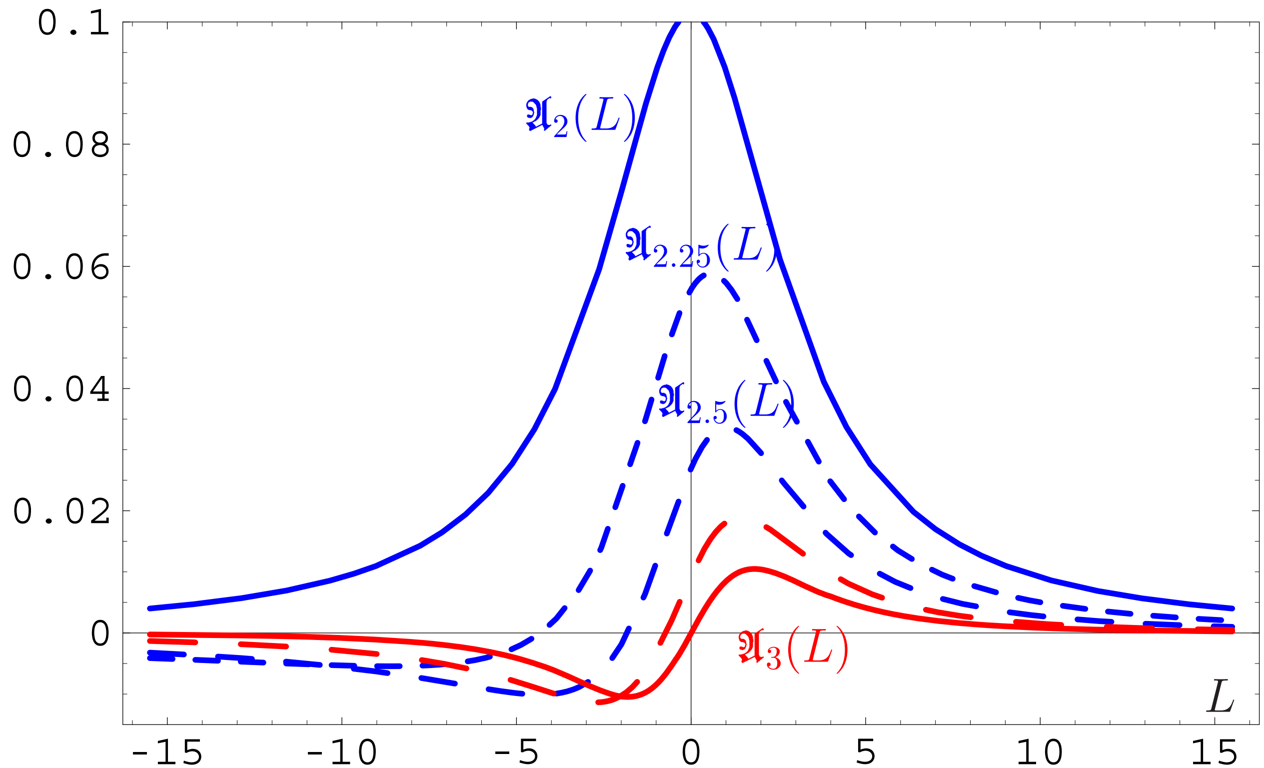
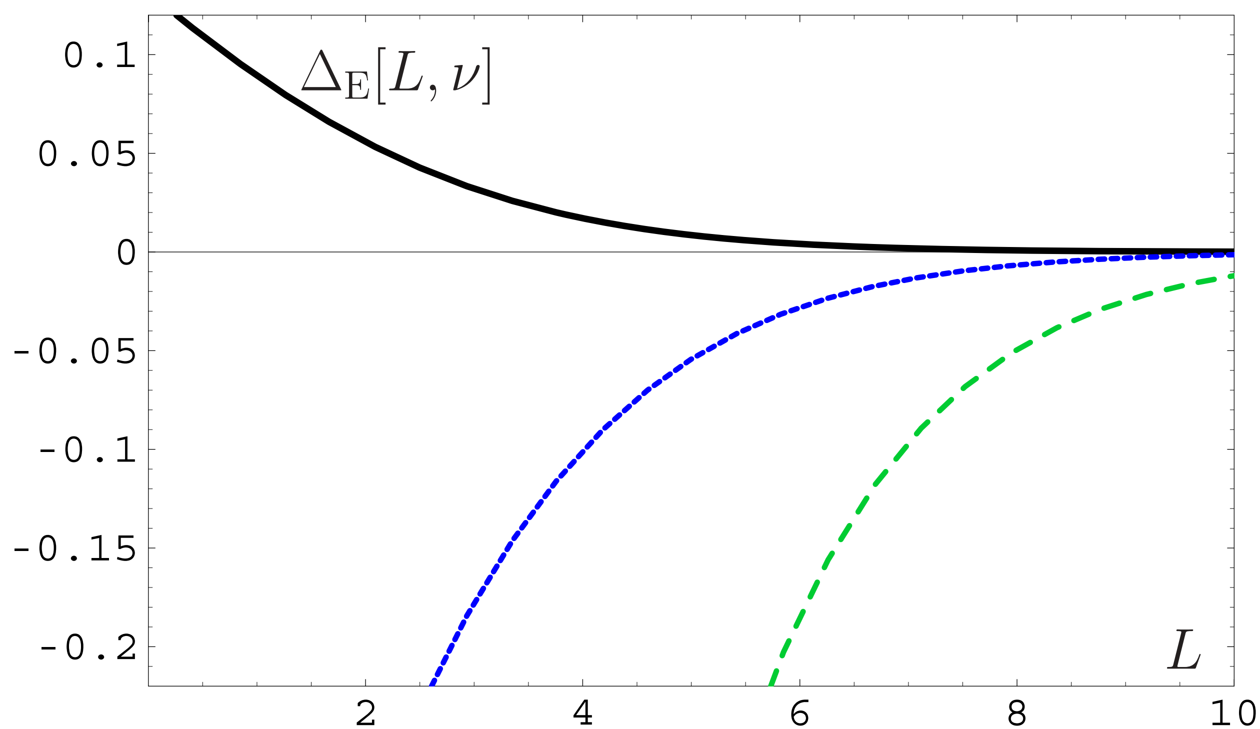
Compare with graphics in Minkowskian region :



FAPT(E): Comparing $\mathcal{A}_\nu$ *with* $(\mathcal{A}_1)^\nu$

$$\boxed{\Delta_{\rm E}(L,\nu) = \frac{\mathcal{A}_\nu[L] - \left(\mathcal{A}_1[L]\right)^\nu}{\mathcal{A}_\nu[L]}}$$

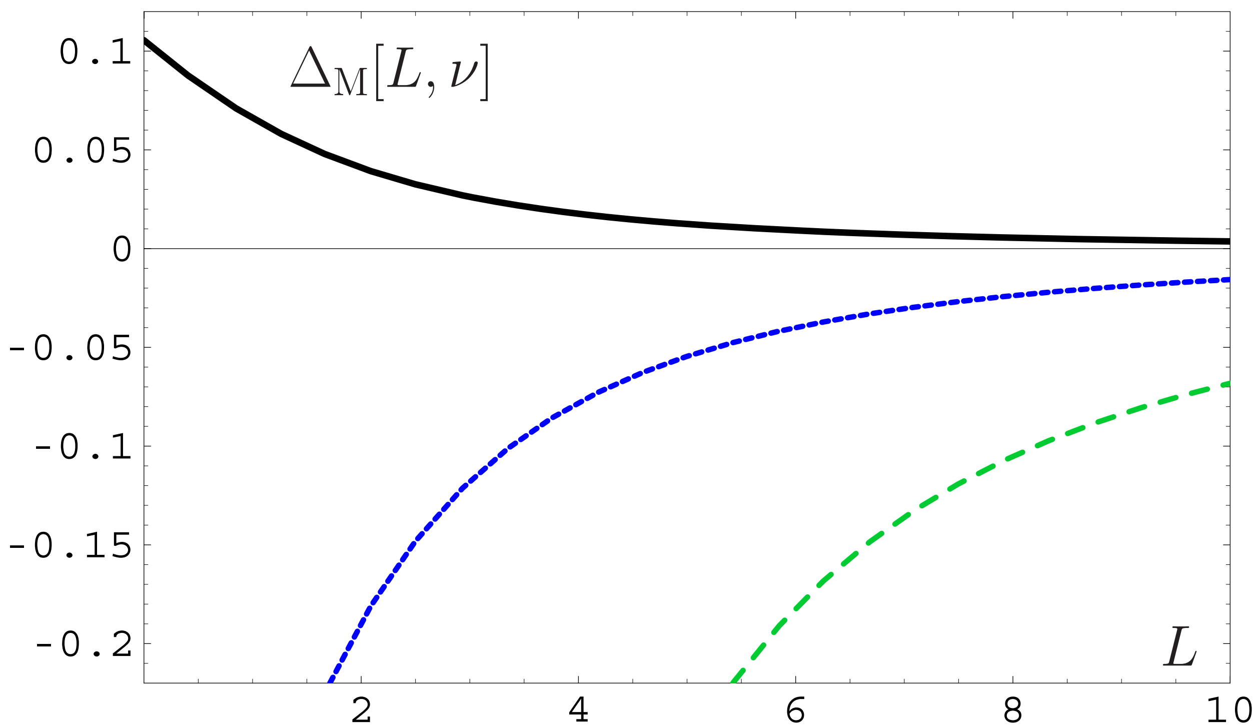
Graphics for fractional $\nu =$**0.62**, **1.62** and **2.62**:



FAPT(M): Comparing $\mathfrak{A}_\nu$ *with* $\left(\mathfrak{A}_1\right)^\nu$

$$\Delta_{\mathrm{M}}(L,\nu) = \frac{\mathfrak{A}_\nu[L] - \left(\mathfrak{A}_1[L]\right)^\nu}{\mathfrak{A}_\nu[L]}$$

Minkowskian graphics for $\nu =$**0.62**, **1.62** and **2.62**:



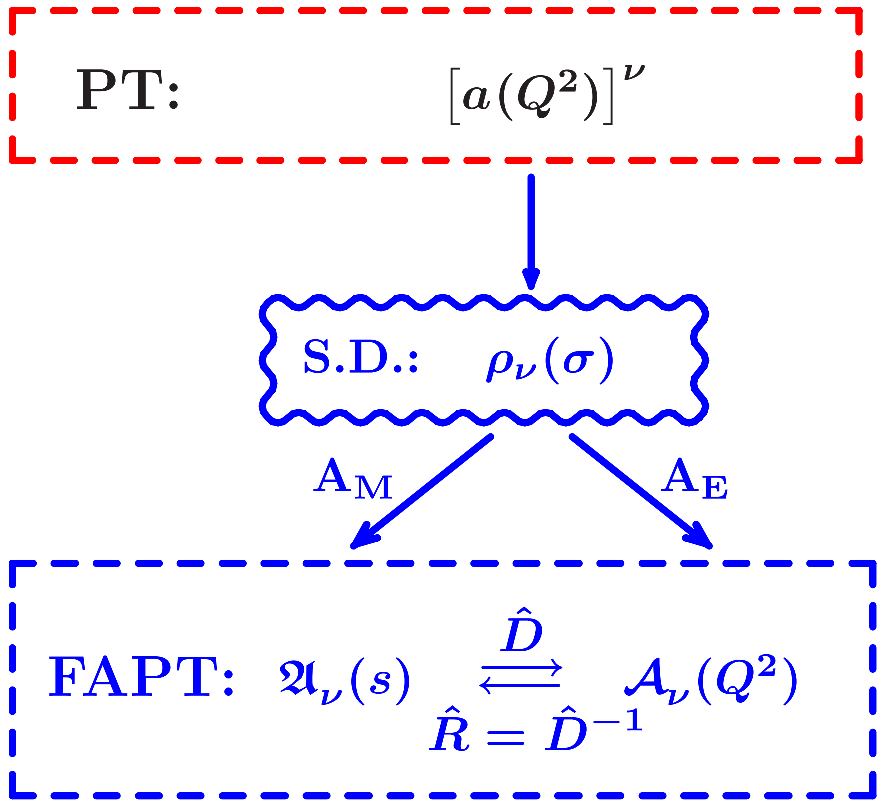
Comparison of PT, APT, and FAPT

Theory	PT	APT	FAPT
Set	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m, \mathfrak{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_\nu, \mathfrak{A}_\nu\right\}_{\nu\in\mathbb{R}}$
Series	$\sum\limits_m f_m\, a^m$	$\sum\limits_m f_m\, \mathcal{A}_m$	$\sum\limits_m f_m\, \mathcal{A}_m$
Inv. powers	$(a[L])^{-m}$	—	$\mathcal{A}_{-m}[L] = L^m$
Products	$a^\mu a^\nu = a^{\mu+\nu}$	—	—
Index deriv.	$a^\nu \ln^k a$	—	$\mathcal{D}^k \mathcal{A}_\nu$
Logarithms	$a^\nu L^k$	—	$\mathcal{A}_{\nu-k}$

Development of FAPT:

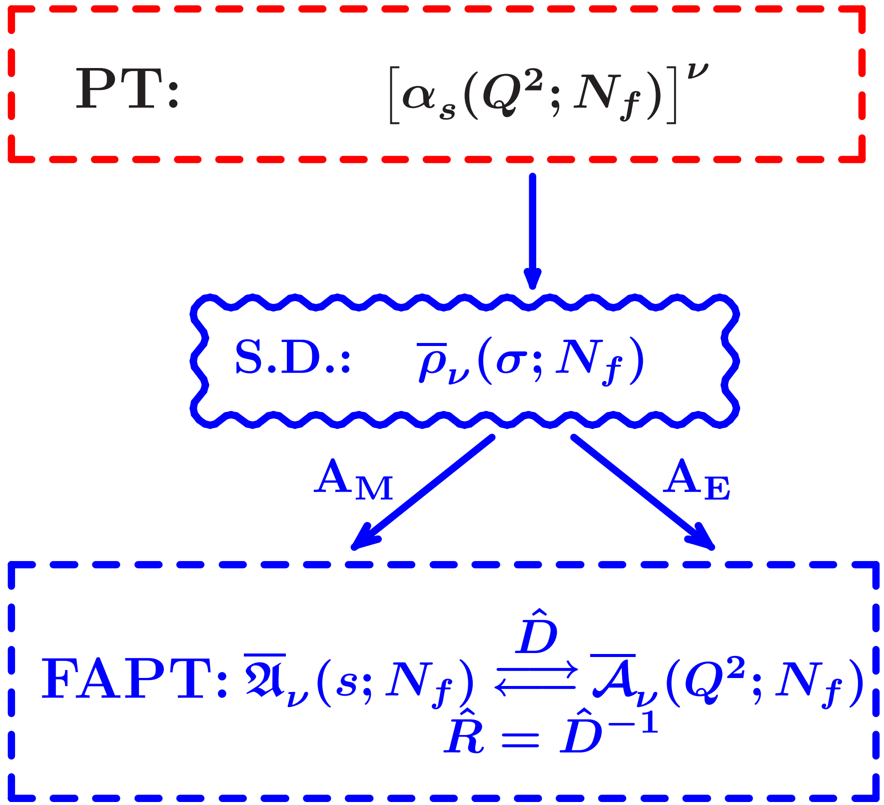
Heavy-Quark Thresholds

*Conceptual scheme of **FAPT***



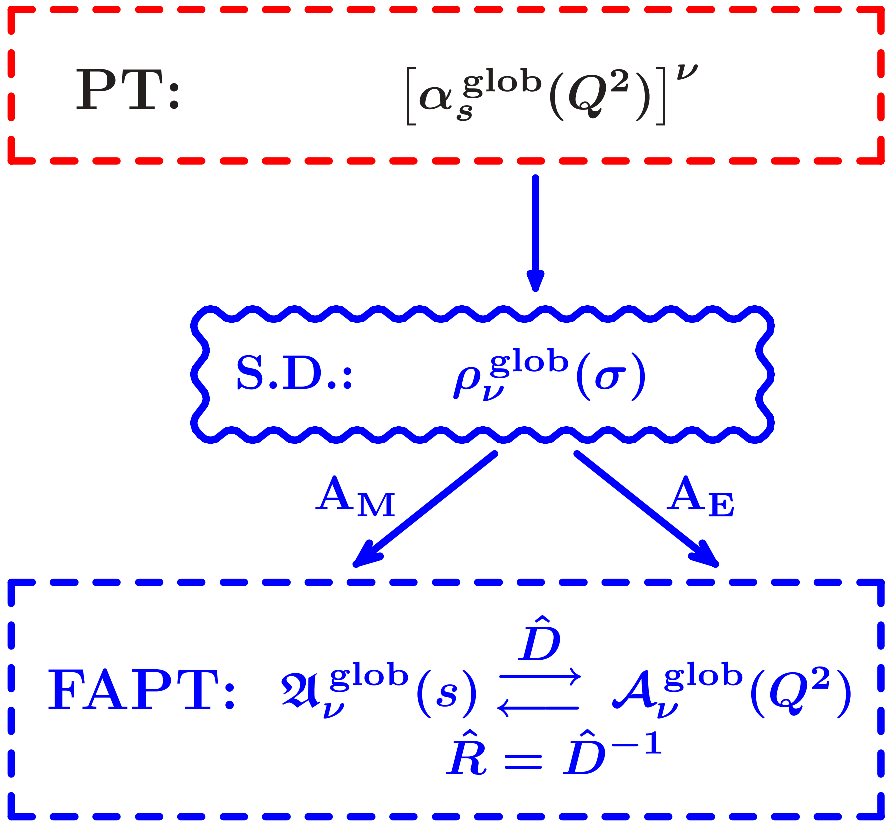
Here N_f is fixed and factorized out.

Conceptual scheme of ***FAPT***



Here N_f is fixed, but not factorized out.

Conceptual scheme of ***FAPT***



Here we see how “analytization” takes into account N_f-dependence.

Global FAPT: Single threshold case

- Consider for simplicity only one threshold at $s = m_c^2$ with transition $N_f = 3 \to N_f = 4$.
- Denote: $L_4 = \ln\left(m_c^2/\Lambda_3^2\right)$ and $\lambda_4 = \ln\left(\Lambda_3^2/\Lambda_4^2\right)$.

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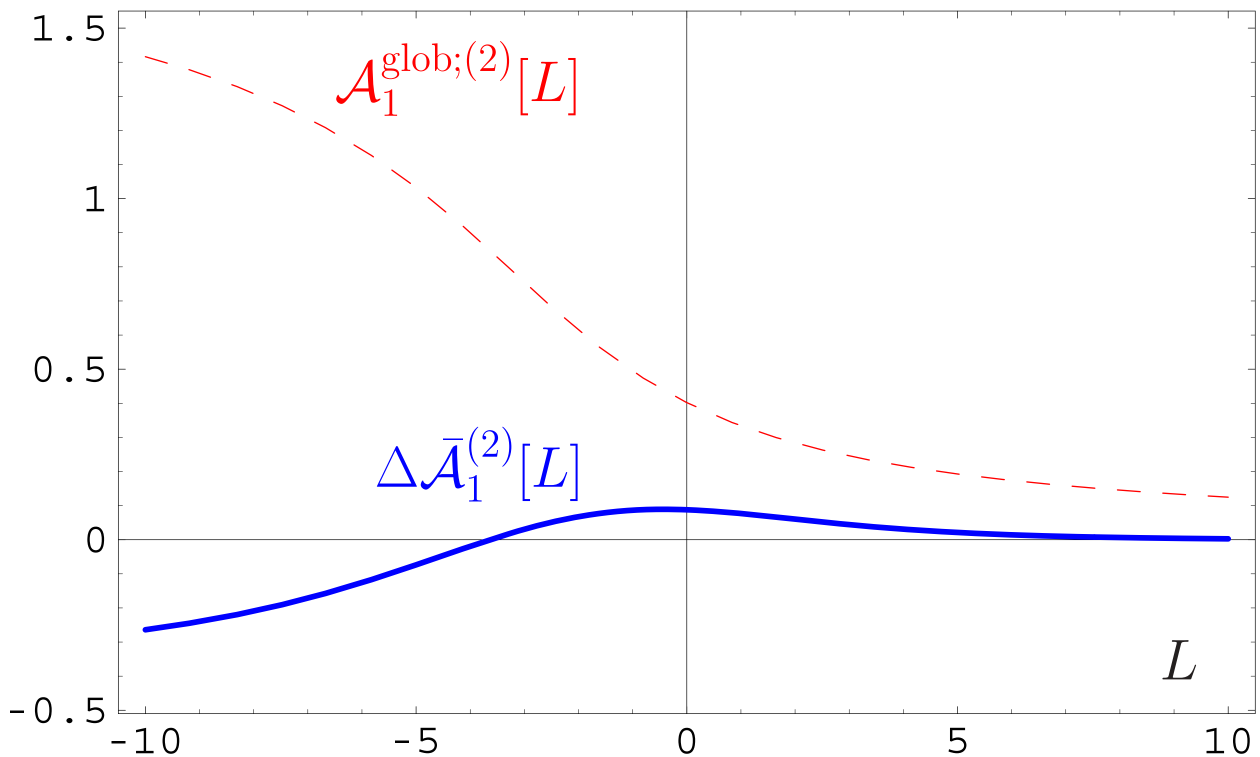
and

$$\mathcal{A}_\nu^{\text{glob}}[L] = \overline{\mathcal{A}}_\nu[L+\lambda_4;4] + \int\limits_{-\infty}^{L_4} \frac{\overline{\rho}_\nu\left[L_\sigma;3\right] - \overline{\rho}_\nu\left[L_\sigma+\lambda_4;4\right]}{1+e^{L-L_\sigma}} dL_\sigma$$

Graphical comparison: Fixed-N_f—***Global***

$$\boxed{\mathcal{A}_{\nu}^{\textbf{glob}}[L] = \overline{\mathcal{A}}_{\nu}[L+\lambda_4;4] + \Delta\overline{\mathcal{A}}_{\nu}[L]\,;}$$

$\Delta\overline{\mathcal{A}}_1[L]$ — **solid**, $\mathcal{A}_1^{\textbf{glob}}[L]$ — **dashed**:



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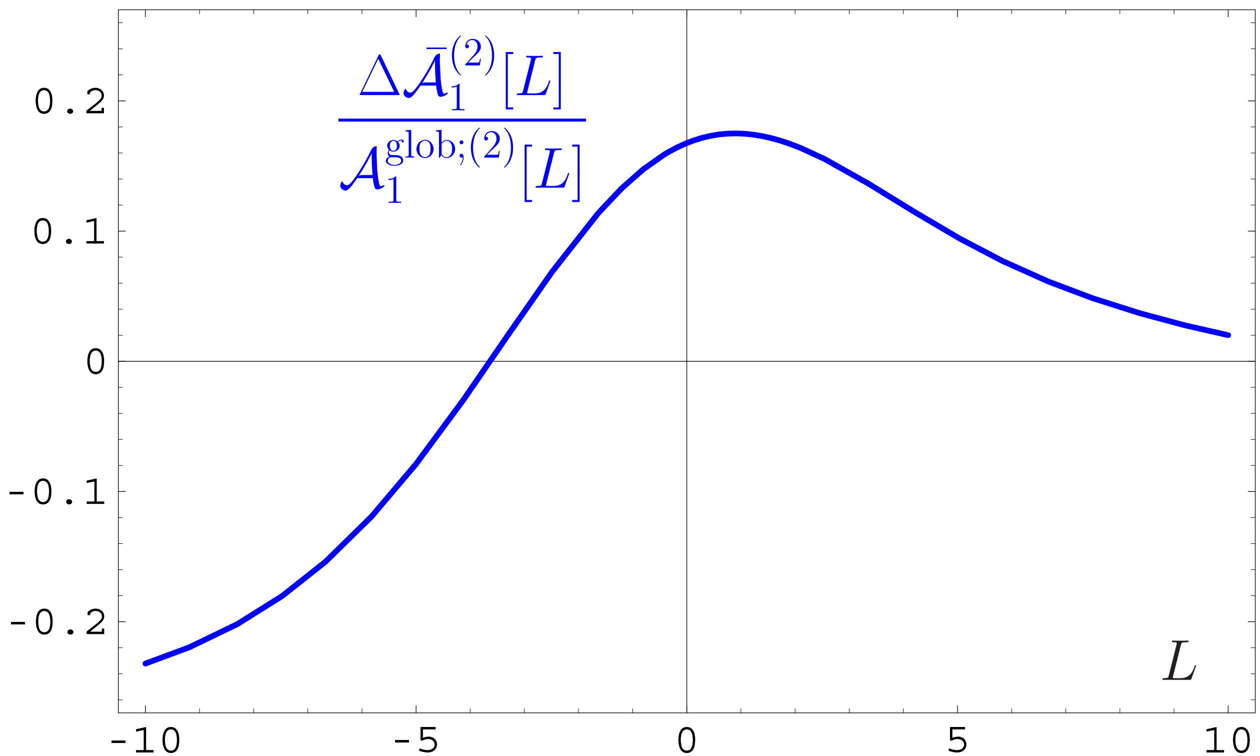

Resummation in one-loop APT and FAPT

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Consider series $\mathcal{D}[L] = d_0 + \sum_{n=1}^{\infty} d_n\,\mathcal{A}_n[L]$

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Let exist the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^\infty P(t)\, t^{n-1} dt \quad \text{with} \quad \int_0^\infty P(t)\, dt = 1\,.$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^\infty f(t)\, P(t)\, dt\,.$$

Then coefficients $d_n = d_1 \, \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

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We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{\Gamma(n+1)} \left(-\frac{d}{dL}\right)^n \mathcal{A}_1[L]\,.$$

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Result:

$$\mathcal{D}[L] = d_0 + d_1 \, \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \, \langle\langle \mathfrak{A}_1[L-t] \rangle\rangle_{P(t)}$$

Resummation in Global Minkowskian APT

Consider series $\mathcal{R}[L] = d_0 + \sum_{n=1}^{\infty} d_n\, \mathfrak{A}_n^{\text{glob}}[L]$

with coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

Result:

$$\mathcal{R}[L] = d_0 \;+\; d_1 \langle\langle \theta\,(L < L_4)\Big[\Delta_4 \overline{\mathfrak{A}}_1[t] + \overline{\mathfrak{A}}_1\Big[L - \frac{t}{\beta_3}; 3\Big]\Big] \rangle\rangle_{P(t)}$$
$$+\; d_1 \langle\langle \theta\,(L \geq L_4) \overline{\mathfrak{A}}_1\Big[L + \lambda_4 - \frac{t}{\beta_4}; 4\Big] \rangle\rangle_{P(t)} \,.$$

where

$$\Delta_4 \overline{\mathfrak{A}}_1[t] = \overline{\mathfrak{A}}_1\Big[L_4 + \lambda_4 - \frac{t}{\beta_4}; 4\Big] - \overline{\mathfrak{A}}_1\Big[L_3 - \frac{t}{\beta_3}; 3\Big] \,.$$

Resummation in Global Euclidean APT

In Euclidean domain the result is more complicated:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \int\limits_{-\infty}^{L_4} \frac{\bar{\rho}_1 \left[L_\sigma; 3\right] \, dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_3}} \rangle\rangle_{P(t)}$$

$$+ \langle\langle \Delta_4[L, t] \rangle\rangle_{P(t)} + d_1 \langle\langle \int\limits_{L_4}^{\infty} \frac{\bar{\rho}_1 \left[L_\sigma + \lambda_4; 4\right] \, dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_4}} \rangle\rangle_{P(t)} \,.$$

where

$$\Delta_4[L, t] = \int\limits_0^1 \frac{\bar{\rho}_1 \left[L_4 + \lambda_4 - tx/\beta_4; 4\right] \, t}{\beta_4 \, \left[1 + e^{L - L_4 - t\bar{x}/\beta_4}\right]} \, dx$$

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with coefficients $d_n = d_1\,\langle\langle t^{n-1}\rangle\rangle_{P(t)}$.

Result:

$$\begin{aligned}
\mathcal{R}_\nu[L] &= d_0\,\mathfrak{A}_\nu[L] + d_1\,\langle\langle \mathfrak{A}_{1+\nu}[L-t]\rangle\rangle_{P_\nu(t)}\,;\\
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\end{aligned}$$

where $P_\nu(t) = \int\limits_0^1 P\left(\frac{t}{1-z}\right)\nu\,z^{\nu-1}\frac{dz}{1-z}\,.$

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Then result is complete analog of the Global APT(M) result with natural substitutions:

$$\overline{\mathfrak{A}}_1[L] \to \overline{\mathfrak{A}}_{1+\nu}[L] \quad \text{and} \quad P(t) \to P_\nu(t)$$

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Higgs boson decay

$$H^0 \to b\bar{b}$$

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\mathsf{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\mathsf{S}}(x) J_{\mathsf{S}}(0)\,]\, |0\rangle$$

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in terms of discontinuity of its imaginary part

$$R_{\mathsf{S}}(s) = \mathbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma_{\mathsf{H}\to b\bar{b}}(M_{\mathsf{H}}) = \frac{G_F}{4\sqrt{2}\pi} M_{\mathsf{H}}\, m_b^2(M_{\mathsf{H}})\, R_{\mathsf{S}}(s = M_{\mathsf{H}}^2)\,.$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

$$m^2(Q^2) = \hat{m}^2 \left[\frac{\alpha_s(Q^2)}{\pi}\right]^{\nu_0} \left[1 + \frac{c_1 b_0 \alpha_s(Q^2)}{4\pi^2}\right]^{\nu_1} .$$

with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 14.6$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$.

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In FAPT(M) we obtain

$$\widetilde{\mathcal{R}}_{\mathbf{S}}^{(l);N}[L] = \frac{3\hat{m}^2}{\pi^{\nu_0}} \left[\mathfrak{A}_{\nu_0}^{(l);\text{glob}}[L] + \sum_{m>0}^{N} \frac{d_m^{(l)}}{\pi^m} \mathfrak{A}_{m+\nu_0}^{(l);\text{glob}}[L]\right]$$

Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m/d_1$, with $d_1 = 17/3$.

Model	$\tilde{d}_1$	$\tilde{d}_2$	$\tilde{d}_3$	$\tilde{d}_4$	$\tilde{d}_5$
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"PMS" model	—	—	64.8	547	7782

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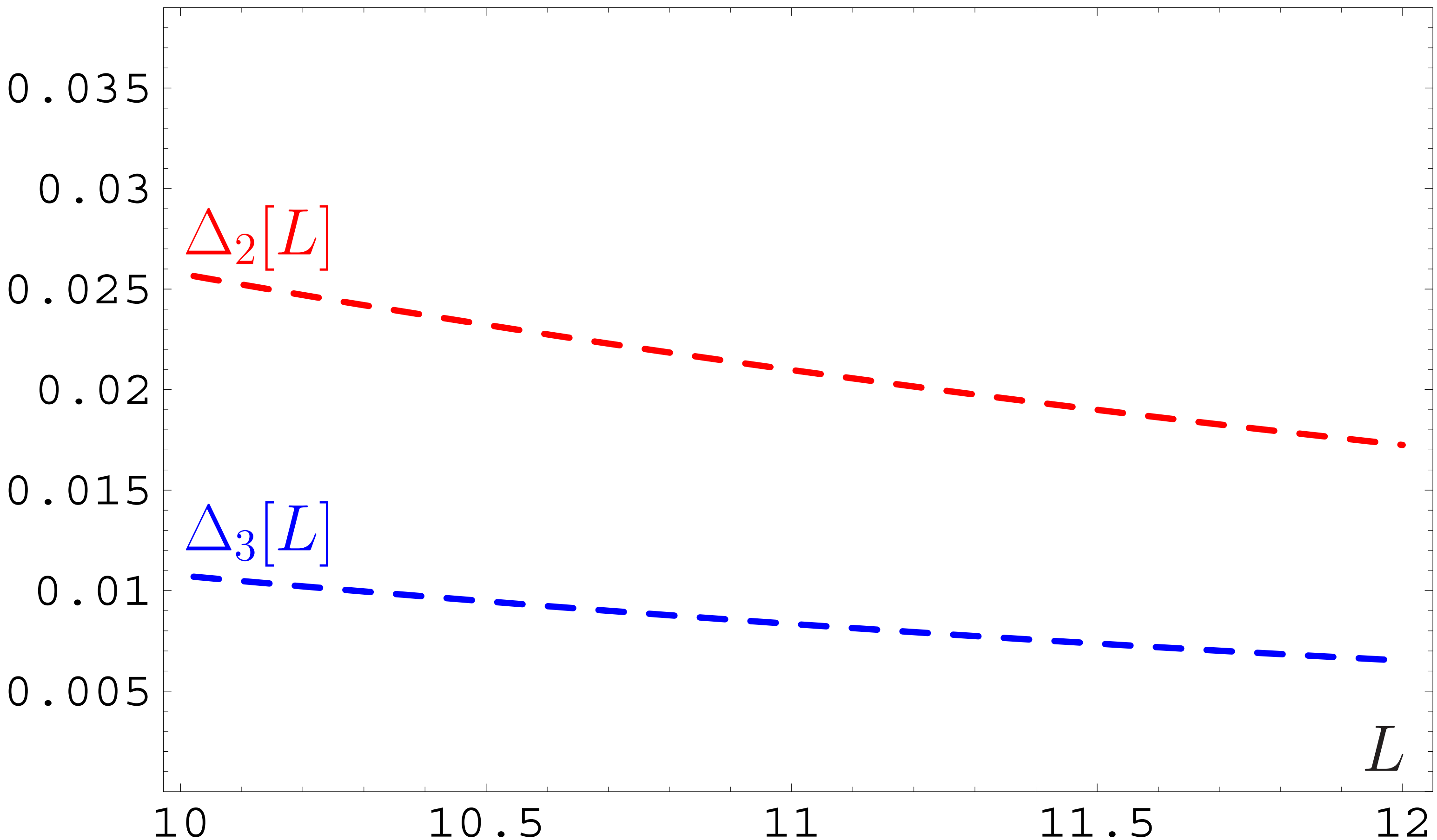
FAPT(M) for $\widetilde{R}_S$: Truncation errors

We define relative errors of series truncation at Nth term:

$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;N)}[L] / \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;\infty)}[L]$$



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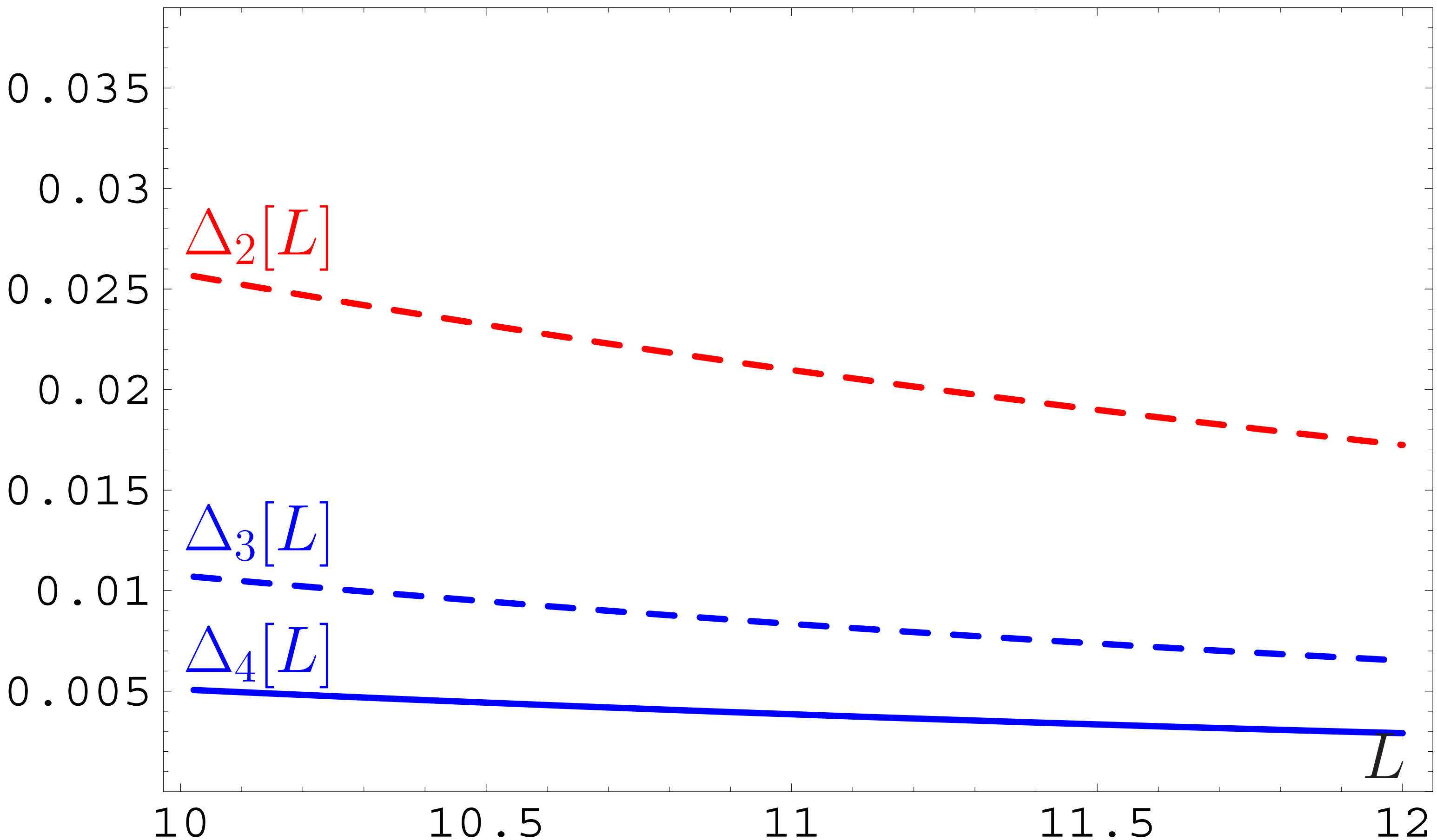

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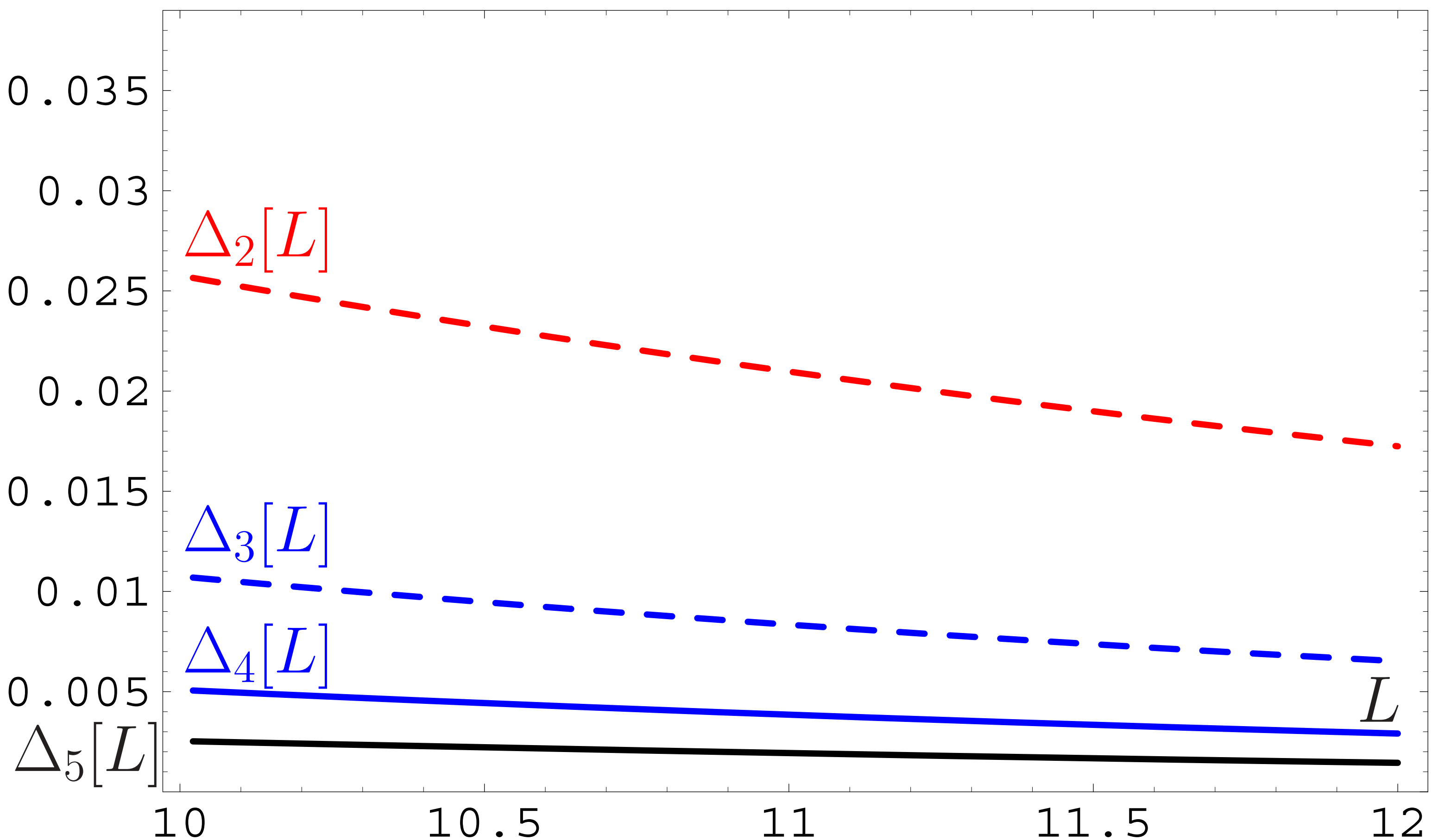

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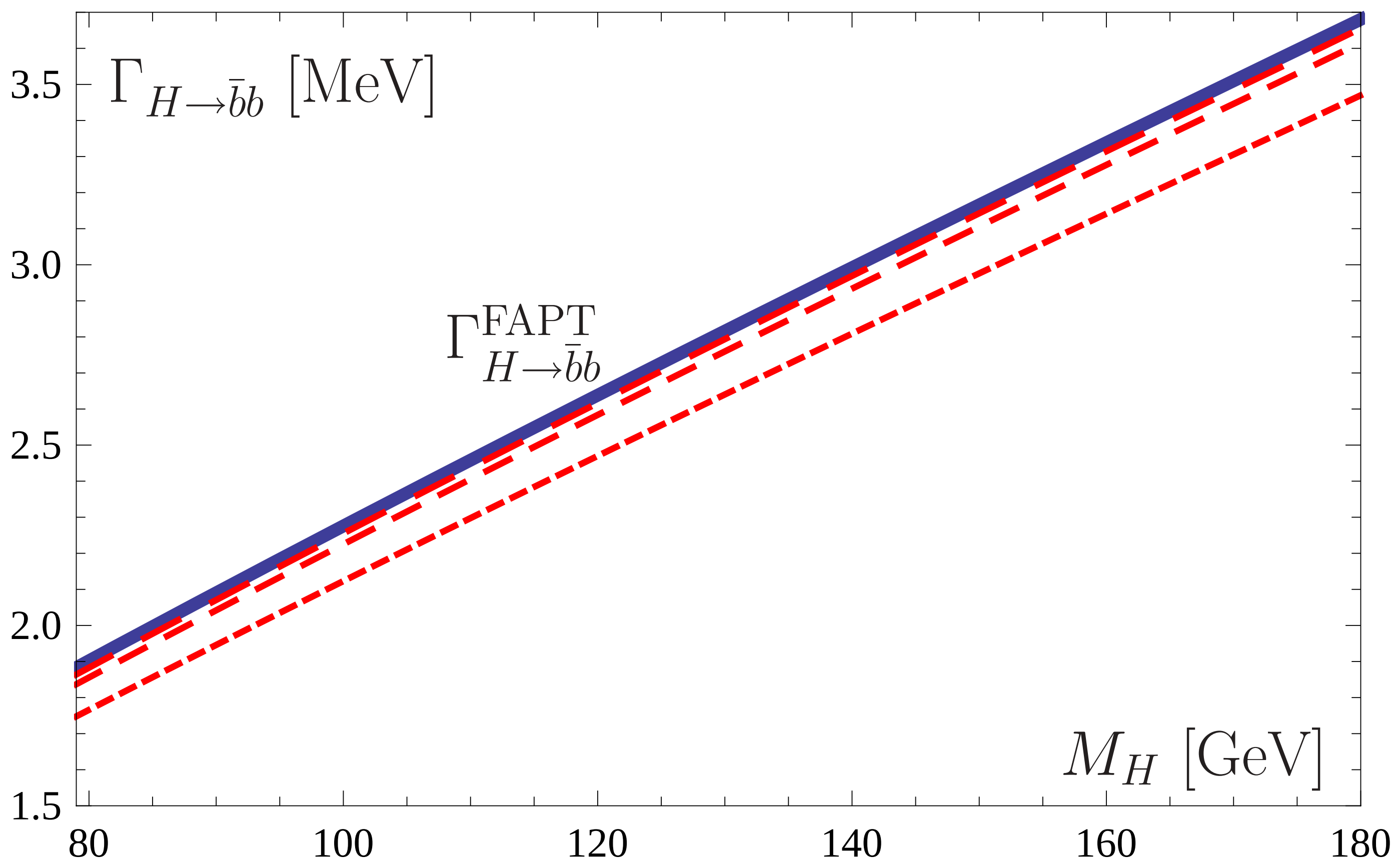
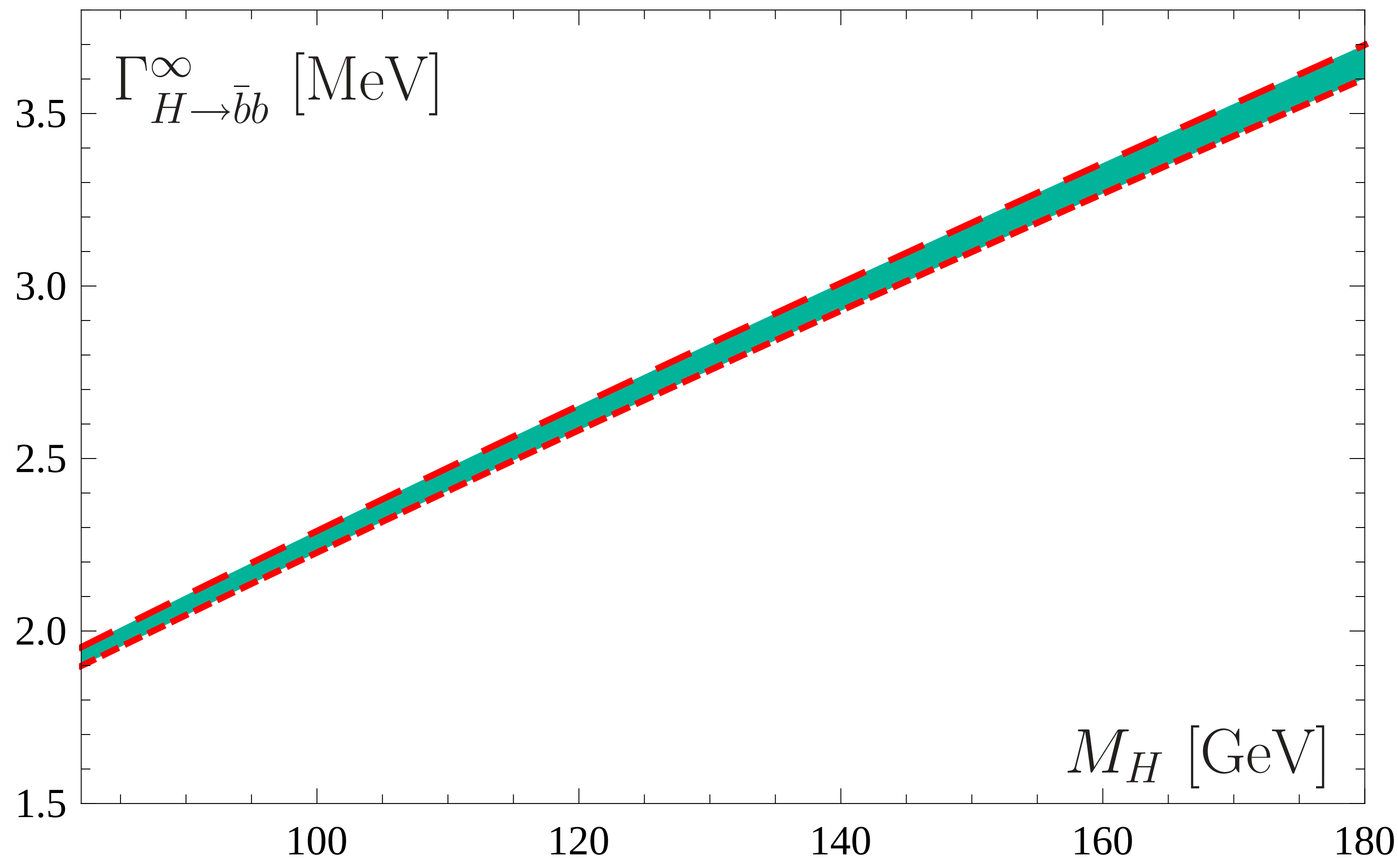
But profit will be tiny — instead of $\mathbf{0.5\%}$ one'll obtain $\mathbf{0.3\%}$!



FAPT(M) for $\widetilde{\boldsymbol{R}}_S$: Truncation errors

Conclusion: If we need accuracy of the order 0.5% — then we need to take into account up to the 4-th correction.

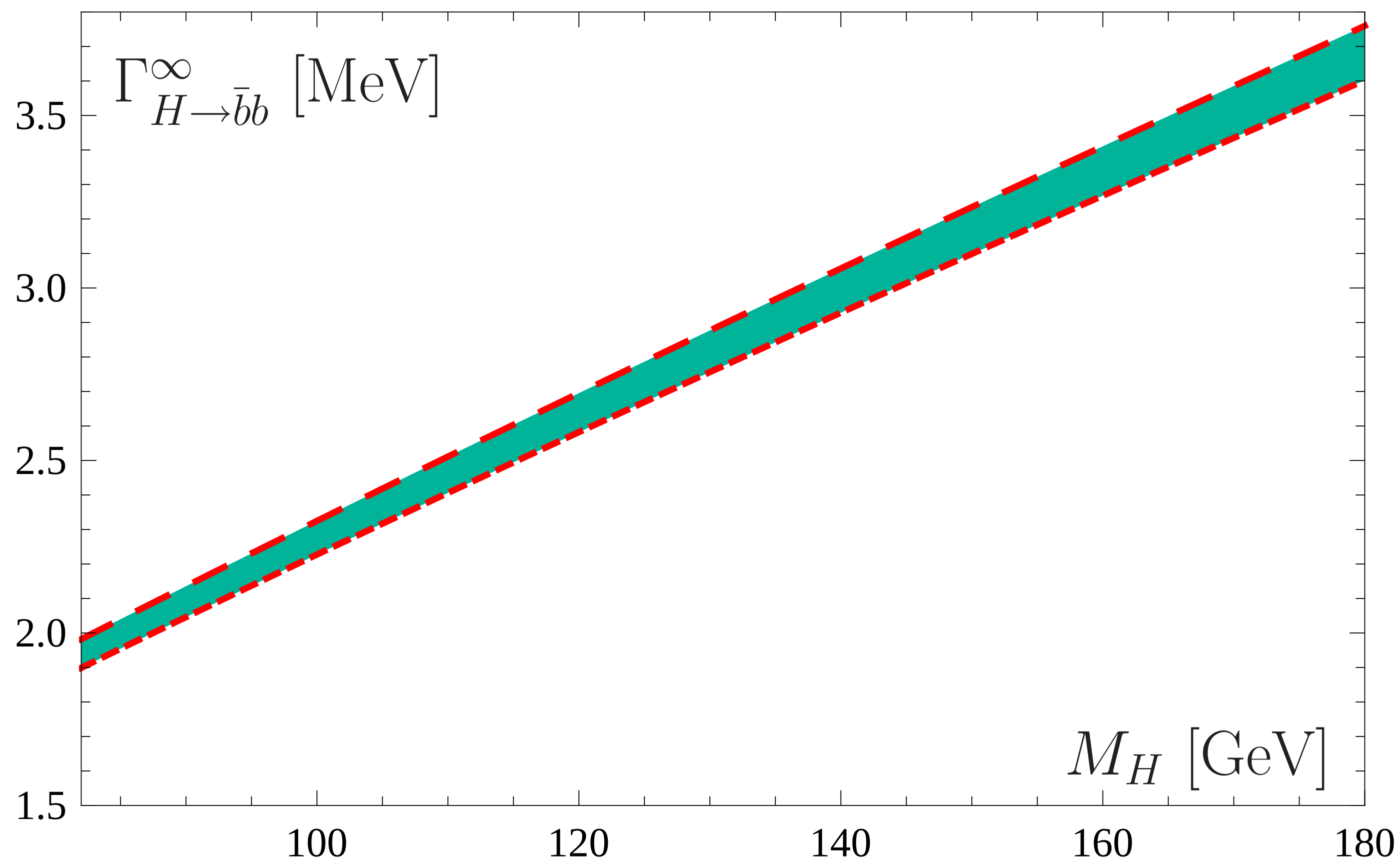
Note: uncertainty due to $\boldsymbol{P(t)}$-modelling is small $\lesssim \mathbf{0.6\%}$.



FAPT(M) for $\widetilde{R}_S$: Truncation errors

Conclusion: If we need accuracy of the order 1% — then we need to take into account up to the 3-rd correction — in agreement with Kataev&Kim **[0902.1442]**.

Note: on-shell mass uncertainty $\sim \mathbf{2}\%$.



FAPT(M) for $\widetilde{R}_S$: Truncation errors

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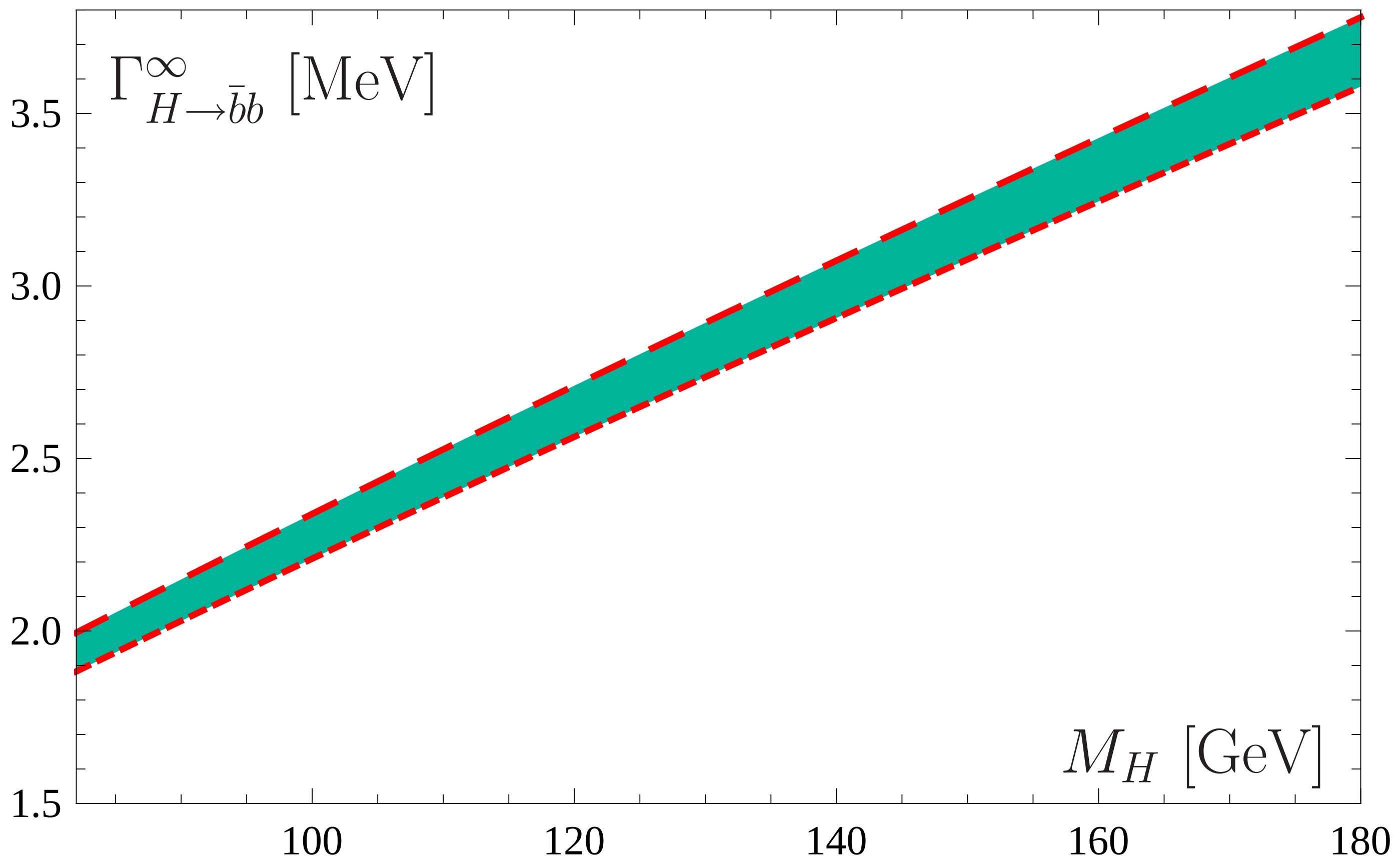
Note: overall uncertainty $\sim 3\%$.



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- Using quite simple model generating function $P(t)$ for Higgs boson decay $H \to \bar{b}b$ we see that at **N^{3}LO** we have accuracy of the order of:
  1% — due to truncation error ;
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  Agreement with Kataev&Kim **[0902.1442]**.