RADIATION OF FORCE-MOMENTUM FROM RELATIVISTIC CHARGED PARTICLES

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1 Introduction

- In 1897 prominent Russian physicist, professor of Imperial University of Tartu A. I. Sadowsky, put forward a hypothesis that circularly polarized light must possess the proper angular momentum [1]. He also proposed a method of measurement of angular momentum of light based on the light transmission through an anisotropic crystalline plate.

- In 1935-1936 B. A. Beth in USA [2] and A. N. S. Holborn in England [3] experimentally proved that circularly polarized light does possess angular momentum.

- [1] A. I. Sadowsky, Zhurn. Russ. Fiz.-Khim. Ob.-va, Fiz. I, Vyp. 2, 82 (1987), in Russian; see also Acta et Comment. Imp. Univ. Jurien. 7, No. 1-2 (1899), in French.
- [2] B. A. Beth, Phys. Rev. 48, 471 (1935); 50, 115 (1936).
- [3] A. N. S. Holborn, Nature, 137,31 (1936).

- At the present time there is no doubt about the existence of the angular momentum of circularly polarized electromagnetic waves.

- However, the general definition of angular momentum of the electromagnetic field (AMEF) is argued over among physicists even nowadays. Contentious debates about the adequacy of the theory of angular momentum and its radiation arise from time to time [4-9].

- [4] G. V. Rosenberg, Usp. Fiz. Nauk, 40, 328 (1950).
- [5] H. C. Ohanian, Amer. J. Phys. 54, 500 (1986).
- [6] K. S. Wulfson, Usp. Fiz. Nauk, 152,668 (1987).
- [7] I. V. Sokolov, Usp. Fiz. Nauk, 161,175 (1991).
- [8] A. L. Barabanov, Usp. Fiz. Nauk, 163, 75, (1993).
- [9] S. A. Akhmanov, S. Yu. Nikitin. “Fiz. Optica”, (MSU, Moscow) 1998.

2 Two alternative methods of AMEF

- The first method was proposed by Ivanenko and Sokolov [10] on the basis of the principle of least action for the electromagnetic field (see, for example, [11]). This principle leads to the conservation laws of the energy density of AMEF in the differential form (see also [11])

$$\mathcal{D}_\lambda \mathcal{M}^{\mu\nu\lambda} = 0.$$

- Here

$$\mathcal{M}^{\mu\nu\lambda} = «\mathcal{A}^{\mu\nu\lambda}» + «\mathcal{S}^{\mu\nu\lambda}»$$

is the density tensor of the total AMEF split into «orbital» and «spin» parts of AMEF:

$$«\mathcal{A}^{\mu\nu\lambda}» = \frac{1}{c}\left(R^\mu \mathcal{T}^{\nu\lambda} - R^\nu \mathcal{T}^{\mu\lambda}\right), \quad «\mathcal{S}^{\mu\nu\lambda}» = \frac{1}{4\pi c}\left(A^\mu H^{\nu\lambda} - A^\nu H^{\mu\lambda}\right)$$

respectively.

- [10] D. D. Ivanenko and A. A. Sokolov, " Classical Field Theory" (GITTL, Moscow-Leningrad), 1949, in Russian; see also "Klassische Fieldtheorie " (Akademie-Verlag, Berlin) 1953, in German.
- [11] F. Rohrlich, "Classical Charged Particles", (World Scientific, Singapore) 2007.

- In addition,

$$\mathcal{T}^{\nu\lambda} = -\frac{1}{4\pi}\left(\frac{1}{4}g^{\nu\lambda}H_{\alpha\beta}H^{\alpha\beta} + \mathcal{D}^{\nu}A^{\rho}H_{\rho}{}^{\lambda}\right) \neq \mathcal{T}^{\lambda\nu}, \quad \mathcal{D}_{\lambda}\mathcal{T}^{\nu\lambda} = 0,$$

is the canonical tensor of the energy density.

- The purpose of the quotes' presence in our notations is due to the fact that the derivative $\mathcal{D}_{\lambda}«\Lambda^{\mu\nu\lambda}» = \mathcal{T}^{\nu\mu} - \mathcal{T}^{\mu\nu}$ is nonzero. This happens because the canonical tensor $\mathcal{T}^{\mu\nu}$ isn't symmetrical and, therefore, the separation of $\mathcal{M}^{\mu\nu\lambda}$ into «orbital» and «spin» parts isn't relativistically invariant.
- Worse than that, $\mathcal{T}^{\mu\nu}$ isn't gauge invariant, so that $«\mathcal{T}^{\mu\nu\lambda}»$ isn't gauge invariant either and this means that it can't claim to be the real observed value.
- In this connection the definition of the proper AMEF or spin (see [10, 11] et al.) (received wide spread occurrence on the ground definition $«\mathcal{T}^{\mu\nu\lambda}»$)

$$\boldsymbol{\Pi} = \frac{1}{4\pi c}\int [\boldsymbol{E}\boldsymbol{A}]dV$$

is required in additional substantiation.

- Alternative pathway is the density tensor of the total AMEF proposed by C. Teitelboim [12] (see also [13,14])

$$\mathcal{M}^{\mu\nu\lambda} = \frac{1}{c}\left(R^{\mu}\mathcal{E}^{\nu\lambda} - R^{\nu}\mathcal{E}^{\mu\lambda}\right)$$

on the basis of the symmetric density tensor of the electromagnetic field energy

$$\mathcal{E}^{\nu\lambda} = -\frac{1}{4\pi}\left(\frac{1}{4}g^{\nu\lambda}H_{\alpha\beta}H^{\alpha\beta} + H^{\nu\rho}H_{\rho}{}^{\lambda}\right) = \mathcal{E}^{\lambda\nu},$$

which possesses gauge-invariance in contrast to the $\mathcal{T}^{\nu\lambda}$.

- [12] C. Teitelboim, D. Villarroel, Nuovo Cim. 3, 1 (1980).
- [13] Ch. G. van Weert, Physica, 65, 452 (1973).
- [14] C. A. Lopez, D. V. Villarroel, Phys. Rev. 11, 2724 (1975).

- Tensor $\mathcal{E}^{\nu\lambda}$ is calculated according to C. Teitelboim in terms of radiation theory of relativistic particles in which

$$H^{\mu\nu} = \overline{H}^{\mu\nu} + \tilde{H}^{\mu\nu},$$

where $\overline{H}^{\mu\nu}$ is the well-known strength tensor of the attached fields in the near-charge zone and $\tilde{H}^{\mu\nu}$ - is in the long-distance region which is called the radiation zone.

- The four-dimensional vector

$$\tilde{r}^{\mu}(\tau, \tilde{t}) = R^{\mu}(\tilde{t}) - r^{\mu}(\tau)$$

traced from the trajectory radius-vector $r^{\mu}(\tau)$ in proper time τ to the spectator point of four-dimensional vector $R^{\mu}(\tilde{t})$.

- Now the decomposition of the density of total AMEF into orbital and spin parts arises automatically

$$\Lambda^{\mu\nu\lambda} = \frac{1}{c}\left(r^{\mu}\mathcal{E}^{\nu\lambda} - r^{\nu}\mathcal{E}^{\mu\lambda}\right), \quad \Pi^{\mu\nu\lambda} = \frac{1}{c}\left(\tilde{r}^{\mu}\mathcal{E}^{\nu\lambda} - \tilde{r}^{\nu}\mathcal{E}^{\mu\lambda}\right).$$

- Tensor $\mathcal{E}^{\nu\lambda}$ decomposes into three parts in agreement with the general theory of the relativistic radiation (see, for example [15])

$$\mathcal{E}^{\nu\lambda}=\overline{\mathcal{E}}^{\nu\lambda}+\overline{\tilde{\mathcal{E}}}^{\nu\lambda}+\tilde{\mathcal{E}}^{\nu\lambda},$$

where

$$\overline{\mathcal{E}}^{\nu\lambda}=-\frac{1}{4\pi}\frac{e^2c^4}{\left(\tilde{r}_\rho v^\rho\right)^4}\left[c^2\frac{\tilde{r}^\nu\tilde{r}^\lambda}{\left(\tilde{r}_\rho v^\rho\right)^2}+\frac{\tilde{r}^\nu v^\lambda+\tilde{r}^\lambda v^\nu}{\tilde{r}_\rho v^\rho}-\frac{1}{2}g^{\nu\lambda}\right]$$

is the energy density tensor of the convection field in the near-charge zone,

$$\overline{\tilde{\mathcal{E}}}^{\nu\lambda}=-\frac{1}{4\pi}\frac{e^2c^2}{\left(\tilde{r}_\rho v^\rho\right)^3}\left[\frac{2c^2\tilde{r}^\nu\tilde{r}^\lambda\tilde{r}_\alpha w^\alpha}{\left(\tilde{r}_\rho v^\rho\right)^3}+\frac{\tilde{r}^\nu v^\lambda+\tilde{r}^\lambda v^\nu}{\left(\tilde{r}_\rho v^\rho\right)^2}\tilde{r}_\alpha w^\alpha-\frac{\tilde{r}^\nu w^\lambda-\tilde{r}^\lambda w^\nu}{\tilde{r}_\rho v^\rho}\right]$$

is the energy density tensor of the mixed fields,

$$\tilde{\mathcal{E}}^{\nu\lambda}=-\frac{1}{4\pi}\frac{e^2}{\left(\tilde{r}_\rho v^\rho\right)^2}\left[c^2\frac{\left(\tilde{r}_\alpha w^\alpha\right)^2}{\left(\tilde{r}_\rho v^\rho\right)^4}-\frac{w_\alpha w^\alpha}{\left(\tilde{r}_\rho v^\rho\right)^2}\right]\tilde{r}^\nu\tilde{r}^\lambda$$

is the energy density tensor of the radiation field in the wave zone.

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[15] “Synchrotron Radiation Theory and its Development”. In Memory of I. M. Ternov. Ed. V. A. Bordovitsyn (World Scientific, Singapore) 1999.

- In the following we will consider the AMEF only in the wave zone, for which

$$\tilde{\mathcal{J}}^{\mu\nu\lambda} = \frac{1}{c}\left(r^{\mu}\tilde{\mathcal{E}}^{\nu\lambda} - r^{\nu}\tilde{\mathcal{E}}^{\mu\lambda}\right) \sim \frac{1}{\tilde{r}^{2}}, \quad \tilde{\tilde{\mathcal{J}}}^{\mu\nu\lambda} = \frac{1}{c}\left(\tilde{r}^{\mu}\tilde{\tilde{\mathcal{E}}}^{\nu\lambda} - \tilde{r}^{\nu}\tilde{\tilde{\mathcal{E}}}^{\mu\lambda}\right) \sim \frac{1}{\tilde{r}^{2}}.$$

- In terms of the derivative of a retarded function

$$\mathcal{D}^{\lambda} = \frac{\partial}{\partial R^{\lambda}} = \tilde{\partial}^{\lambda} + \frac{\tilde{r}^{\lambda}}{r_{\rho} v^{\rho}} \frac{d}{d\tau}$$

it can be shown that

$$\mathcal{D}_{\lambda} \mathcal{E}^{\nu\lambda} = 0,$$

and also

$$\mathcal{D}_{\lambda}(\overline{\mathcal{E}}^{\nu\lambda} + \tilde{\tilde{\mathcal{E}}}^{\nu\lambda}) = 0, \quad \mathcal{D}_{\lambda}\tilde{\mathcal{E}}^{\nu\lambda} = 0.$$

- From this one can obtain,

$$\mathcal{D}_\lambda \tilde{\mathcal{R}}^{\mu\nu\lambda} = 0, \quad \mathcal{D}_\lambda \tilde{\mathcal{T}}^{\mu\nu\lambda} = 0.$$

and as a consequence

$$\mathcal{D}_\lambda \tilde{\mathcal{M}}^{\mu\nu\lambda} = 0,$$

- Thus, the decomposition of the density of the total AMEF onto orbital and spin parts is now gauge and relativistically invariant. Therefore, we will assume the Teitelboim's method as the basis of our further research. We shall also use the method of D. D. Ivanenko and A. A. Sokolov for the comparison because in accordance with the theory of relativistic radiation in the wave zone

$$\mathcal{T}^{\nu\lambda} = \mathcal{T}^{\lambda\nu}$$

and the previously denoted defects of this method disappear!

3 Relativistic theory of the AMEF radiation

- First, we write expressions for the density tensors of orbital momentum in explicit form with regard to wave zone

$$\tilde{\lambda}^{\mu\nu\lambda} = -\frac{e^2}{4\pi c}\left(w_\lambda w^\lambda + c^2\frac{(\tilde{r}_\lambda w^\lambda)^2}{(\tilde{r}_\rho v^\rho)^2}\right)\frac{(r^\mu \tilde{r}^\nu - r^\nu \tilde{r}^\mu)\tilde{r}^\lambda}{(\tilde{r}_\rho v^\rho)^4},$$

and the density tensor of the spin momentum

$$\tilde{\pi}^{\mu\nu\lambda} = -\frac{e^2 c}{4\pi}\left(\frac{(\tilde{r}^\mu v^\nu - \tilde{r}^\nu v^\mu)\tilde{r}_\lambda w^\lambda}{(\tilde{r}_\rho v^\rho)^2} - \frac{\tilde{r}^\mu w^\nu - \tilde{r}^\nu w^\mu}{\tilde{r}_\rho v^\rho}\right)\frac{\tilde{r}^\lambda}{(\tilde{r}_\rho v^\rho)^3}.$$

- Now we can obtain orbital and spin momenta of the radiation field on the basis of the Gauss's integral theorem in the form

$$\tilde{L}^{\mu\nu} = \oint \lambda^{\mu\nu\lambda} d\sigma_\lambda, \quad \tilde{\Pi}^{\mu\nu} = \oint \pi^{\mu\nu\lambda} d\sigma_\lambda.$$

- Here $d\sigma_\lambda = e_\lambda \varepsilon^2 c\, d\tau\, d\Omega_0$ is an element of spacelike hypersurface with the normal unit vector $e_\lambda = k_\lambda - v_\lambda / c$, and $d\Omega_0$ is the solid angle element in the rest frame of the particle.

- Integrating $\tilde{L}^{\mu\nu}$ and $\tilde{\Pi}^{\mu\nu}$ over the angles in the rest frame we can obtain expressions for the power of the orbital and spin AMEF radiation in the wave zone

$$\frac{d\tilde{L}^{\mu\nu}}{d\tau} = \frac{2}{3}\frac{e^2}{c^5} w_\rho w^\rho \left(r^\mu v^\nu - r^\nu v^\mu\right), \quad \frac{d\tilde{\Pi}^{\mu\nu}}{d\tau} = \frac{2}{3}\frac{e^2}{c^3}\left(v^\mu w^\nu - v^\nu w^\mu\right).$$

- These formulas have clear-cut physical interpretation. If we put the well-known charge power of the radiation into operation [15]

$$\frac{d\tilde{P}^\mu}{d\tau} = \frac{2}{3}\frac{e^2}{c^5} w_\rho w^\rho v^\mu,$$

we can get the expression for the orbital momentum of radiation,

$$\frac{d\tilde{L}^{\mu\nu}}{d\tau} = r^\mu \frac{d\tilde{P}^\nu}{d\tau} - r^\nu \frac{d\tilde{P}^\mu}{d\tau} = \frac{d}{d\tau}\left(r^\mu \tilde{P}^\nu - r^\nu \tilde{P}^\mu\right),$$

which is exactly same relation between the torque tensor of the particle and its angular momentum tensor derivative as in the relativistic mechanics

$$\tilde{T}^{\mu\nu} = r^\mu \tilde{F}^\nu - r^\nu \tilde{F}^\mu = \frac{d\tilde{L}^{\mu\nu}}{d\tau}.$$

- Thus, the radiation of the orbital AMEF adds up to the radiation of the field momentum of the force.
- The power of the radiation of the spin momentum is proportional to the Thomas's precession frequency, $\Omega^{\mu\nu}_{Th}$ which is defined by this equation in the relativistic spin theory (see [15])

$$\left(\frac{d\pi^{\mu}}{d\tau}\right)_{Th} = \frac{1}{c^2}\left(v^{\mu}w^{\nu} - v^{\nu}w^{\mu}\right)\pi_{\nu} = \Omega^{\mu\nu}_{Th}\,\pi_{\nu},$$

where π^{μ} is four-dimensional space-like spin vector. If we put field momentum of force for the spin radiation into operation similarly to $\tilde{T}^{\mu\nu}$

$$\tilde{G}^{\mu\nu} = \frac{d\tilde{\Pi}^{\mu\nu}}{d\tau},$$

then Thomas's precession (which until now was known as purely kinematic relativistic effect) becomes obvious dynamic interpretation

$$\frac{2}{3}\frac{e^2}{c}\left(\frac{d\pi^{\mu}}{d\tau}\right)_{Th} = \tilde{G}^{\mu\nu}\pi_{\nu}.$$

Conclusion

- Thus, in this presentation two alternative definitions of density of intrinsic angular momentum of electromagnetic field by Ivanenko-Sokolov and by Teitelboim are discussed. It is shown that in spite of the absence of the outward sign of similarity both definitions give identical integral characteristics of radiation for an arbitrary moving relativistic charge. The total power of orbital momentum radiation is proportional to the relativistic angular momentum of the very particle, and corresponds to the radiation torque. The spin angular momentum of radiation is proportional to the kinematical Thomas spin precession, which becomes obvious dynamical interpretation.
- To sum up, it can be stated that this work has discovered a new trend in the relativistic theory of radiation and the spin properties of relativistic particles.

Thank you for your attention!