On the global geometry of Brane Universe models

A. Smirnov V. Berezin

Department of Theoretical Physics
Institute for Nuclear Research

14-th Lomonosov Conference on Elementary Partcile Physics, 2009

Aim

Investigate possible global geometries of “brane universe” scenario. Our Universe is a thin shell/membrane/brane - embedded into the space-time of larger number of dimensions (“bulk”).

Strategy

Simplify everything as much as possible and construct some exactly solvable model - the source of our physical intuition.

Model

(N+1)-dimensional space-time contains N-dimensional brane (= thin shell). Brane is time-like and have have the “cosmological symmetry” (homogeneity + isotropy).

Simplification: step I

Outside the shell - bulk geometry posseses some symmetry: bulk does not depend on the brane position $\Rightarrow$ everywhere in the normal Gaussian coordinate system

$$ds^2 = -dn^2 + B^2(n,t)dt^2 - A^2(n,t)dl^2_{N-1},$$

where

$$dl^2_{N-1} = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{N-2}$$

is the Robertson-Walker line element. As usual
$k = +1$ - (N-1)-dimensional sphere
$k = -1$ - rotational hyperboloid
$k = 0$ - flat space. Energy-momentum tensor

$$T_{\mu\nu} = S_{\mu\nu}\delta(n) + [T_{\mu\nu}]\,\Theta(n) + T_{-\mu\nu}$$

$S_{\mu\nu}$ - surface energy-momentum tensor of the brane ($n = 0$).
Notations: $[X] = X_+ - X_-$, $n > 0$ (“+”), $n < 0$ (“-”)

Local approach (connected to the brane)

Decomposition of Einstein equations

$$\begin{aligned} -([K_{ik}] - \gamma_{ik}[K]) &= 8\pi G S_{ik}, \\ {}^{(N)}G_{ik} &= 8\pi G T_{-ik} + 8\pi G T^{ind}_{-ik}, \end{aligned}$$

$$T^{ind}_{-ik} = -(K_{-ik,n} - \gamma_{ik} K_{-,n} + 2K_{-il} K^{l}_{-k} - K_{-ik} K + \frac{1}{2}\gamma_{ik}(K^{l}_{-j} K^{j}_{-l} + K)).$$

Cosmological time and scale factor on the brane

$$d\tau = B(n,t)dt, \quad a(\tau) = A(n,t)$$

$$\begin{aligned} {}^{(N)}G^0_0 &= \frac{(N-1)(N-2)}{2}\frac{a_\tau^2 + k}{a^2}, \\ {}^{(N)}G^2_2 &= \frac{N-2}{2}\left(\frac{2a_{\tau\tau}}{a} + (N-3)\frac{a_\tau^2 + k}{a^2}\right), \end{aligned}$$

Note, by symmetry ${}^{(N)}G^2_2 = {}^{(N)}G^3_3 = \ldots = {}^{(N)}G^N_N$.

Global approach

Cosmological symmetry allows us to deals with the invariants of bulk geometry only. Namely (d+2)-decomposition (d+2 = N+1)

$$ds^2 = \gamma_{AB}(x)dx^A dx^B - R^2(x)dl_d^2, \quad A, B = 0, 1,$$

dl_d^2 - Robertson-Walker, curvature $d(d-1)k$, $k = \pm 1, 0$, R - scale factor = radius. Two invariants

$$R(t,q), \quad \Delta = \gamma^{AB} R_{,A} R_{,B}$$

Einstein equations + integrability condition

$$\begin{aligned}
\left(R^{d-1}(\Delta + k)\right)_{,A} &= \frac{16\pi G}{d} R^d \left(T R_{,A} - T_A^B R_{,B}\right) \\
\gamma^{AC} R_{||CB} &= -\frac{8\pi G}{d} R T_B^A
\end{aligned}$$

R,T-regions

Δ, R allow us to classify regions of curved space-time.

- R-regions: $\Delta < 0 \Rightarrow R$=const, R can be chosen as a spatial coordinate. Sign $R_{,q}$ cannot be changed.

$$\begin{array}{ll} \text{R}_{+}\text{-region if} & R_{,q} > 0 \\ \text{R}_{-}\text{-region if} & R_{,q} < 0 \end{array}$$

- T-regions: $\Delta > 0 \Rightarrow R$=const, R can be chosen as a time coordinate. Sign $R_{,t}$ cannot be changed.

$$\begin{array}{ll} \text{T}_{+}\text{-region if} & R_{,t} > 0 \text{ - inevitable expansion} \\ \text{T}_{-}\text{-region if} & R_{,t} < 0 \text{ - inevitable contraction} \end{array}$$

- Apparent horizons $\Delta = 0$

Global geometry = set of R,T-regions and apparent horizons

Simplification: step II

$n \neq 0$ - vacuum

$$\Delta = -k + \frac{2Gm}{R^{N-2}} + \frac{2}{N(N-1)}\Lambda R^2.$$

Different k - different bulk $\Rightarrow$ different global structure

Two-dimensional metric in R-regions,

$$ds_2^2 = (-\Delta)\left(dt^2 - dR^{\star 2}\right), \quad dR^\star = \pm\frac{dR}{|\Delta|},$$

Two-dimensional metric in T-regions,

$$ds_2^2 = \Delta\left(dR^{\star 2} - dq^2\right).$$

Single brane: absence of singularity at $R = 0 \Rightarrow m = 0$

$$\Delta = \frac{1}{B^2(n,t)}R^2(n,t)_{,t} - R^2(n,t)_{,n} = f^2(t) - R^2_{,n},$$

$$R_{,n} = \pm\sqrt{f^2(t) - \Delta} = \sigma\sqrt{f^2(t) - \Delta}, \quad \sigma = \text{sign of R-region}$$

Simplification: step III

$S_0^0 = S_2^2 \Rightarrow S_0^0$=const - vacuum brane. Set of equations

$$
\begin{aligned}
R_{,n}(\pm) &= \sigma_{\pm}\sqrt{f^2(t)+k-\frac{2\Lambda}{N(N-1)}R^2}\,,\\
-\left[\frac{R_{,n}}{R}\right] &= \frac{1}{R}\left(R_{,n}(-)-R_{,n}(+)\right)=\frac{8\pi G}{N-1}S_0^0\,,\\
S_0^0 &= \text{const}\,,\quad R(0,t)=a(\tau)\,,\quad \tau=t\,,\\
\frac{(N-1)(N-2)}{2}\frac{a_\tau^2+k}{a^2} &= \frac{N-2}{N}\left(\Lambda+\frac{N}{2(N-1)}(4\pi G)^2\left(S_0^0\right)^2\right)\,.
\end{aligned}
$$

$$R_{,n}^{+} = -R_{,n}^{+} \Rightarrow \sigma=\sigma_-=-\sigma_+ \Rightarrow \mathbb{Z}_2\text{-symmetry}$$

$$\text{sign}S_0^0 = \sigma$$

Inner evolution on the brane does not depend on σ but the bulk does

Solutions $\Lambda > 0$

$$R = R_0 \sqrt{f^2(t) + k} \sin\left(\frac{\sigma n}{R_0} + \varphi(t)\right), \quad R_0 = \sqrt{\frac{N(N-1)}{2\Lambda}}$$

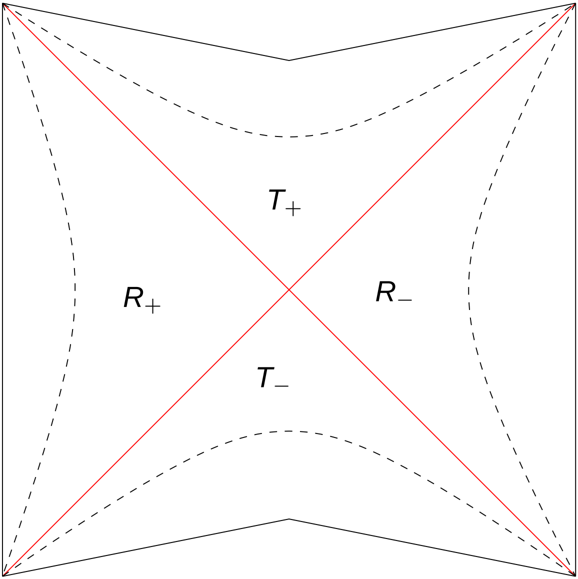
On the brane

$$\begin{aligned} \frac{\sigma}{R_0} \cot\varphi &= \frac{4\pi G}{N-1} S_0^0, \\ \frac{a_\tau^2 + k}{a^2} &= \frac{2\Lambda}{N(N-1)} + \left(\frac{4\pi G}{N-1}\right)^2 \left(S_0^0\right)^2 = \frac{1}{R_0^2 \sin^2\varphi}. \end{aligned}$$

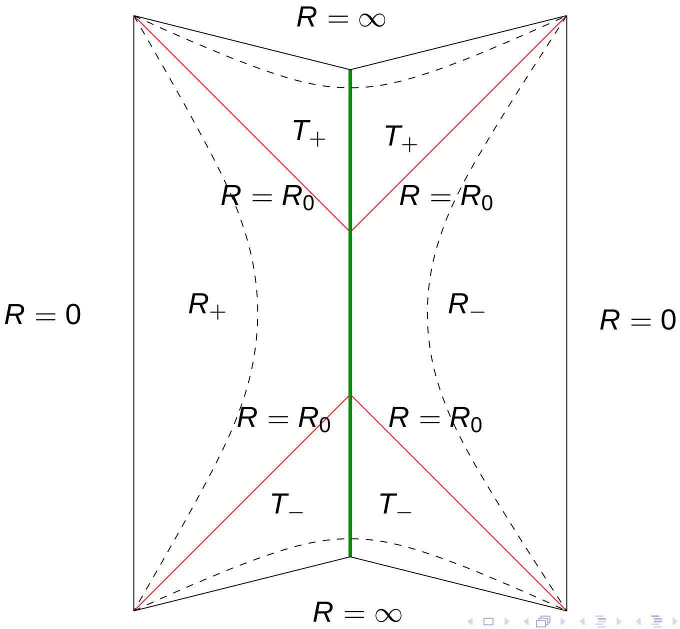
$$\begin{aligned} R &= R_0 \sin\left(\frac{\sigma n}{R_0} + \varphi_0\right) \begin{cases} \cosh\frac{t}{a_0}, & \text{for} \quad k = +1 \\ e^{\frac{t}{a_0}}, & \text{for} \quad k = 0 \\ \left|\sinh\frac{t}{a_0}\right|, & \text{for} \quad k = -1 \end{cases} \\ a_0 &= R_0 \sin\varphi_0 \end{aligned}$$

Different $\sigma \Rightarrow$ different matching

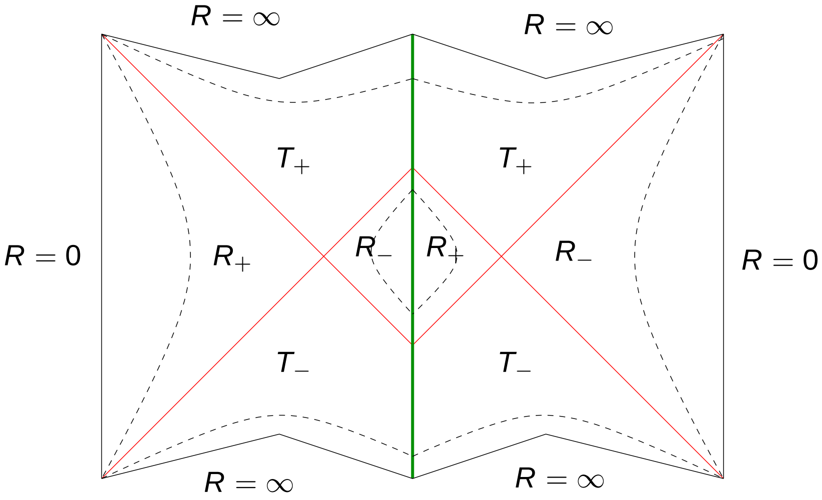
Bulk: Carter-Penrose diagrams $\Lambda > 0$, $k = +1$



$\Lambda > 0, k = +1, S_0^0 > 0$

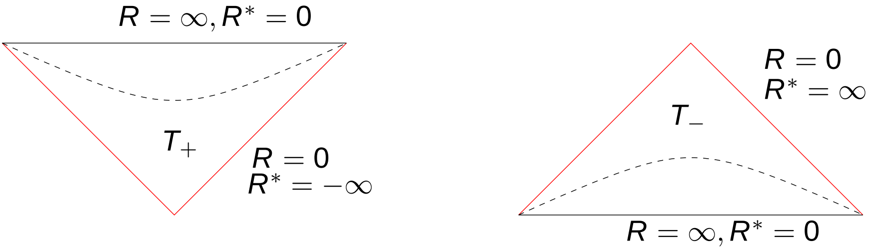


$\Lambda > 0, k = +1, S_0^0 < 0$

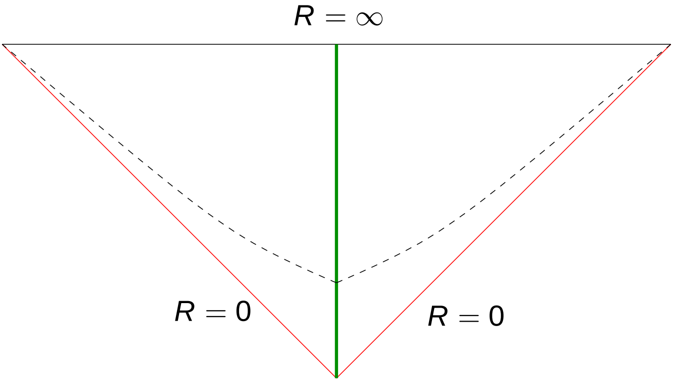


Non-traversable wormholes on both sides

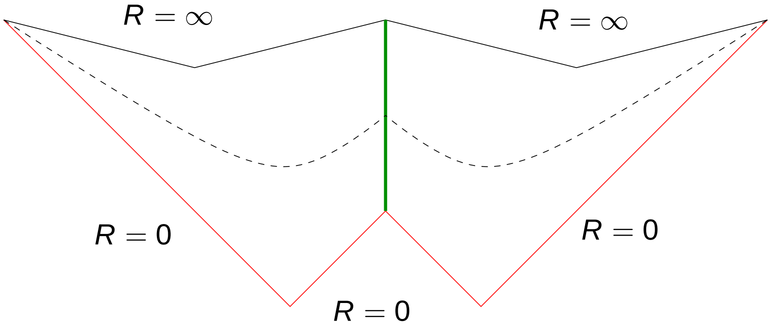
Bulk: Carter-Penrose diagrams $\Lambda > 0$, $k = 0$



$\Lambda > 0, k = 0, S^0_0 > 0$



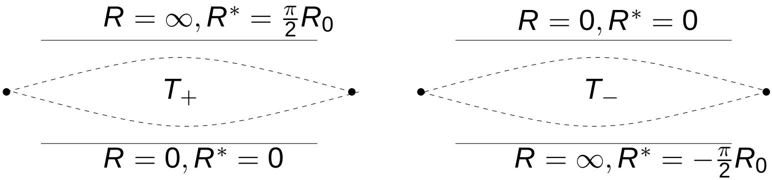
$\Lambda > 0, k = 0, S_0^0 < 0$



Bulk: Carter-Penrose diagrams $\Lambda > 0$, $k = -1$

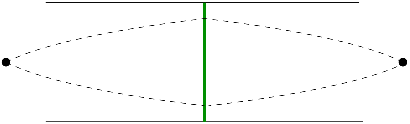
$T_{\pm}$-regions everywhere

$$R^{\star} = \pm \arctan \frac{R}{R_0}\,, \implies R = \pm R_0 \tan \frac{R^{\star}}{R_0}\,,$$



$\Lambda > 0, k = 0, S_0^0 > 0, S_0^0 < 0$

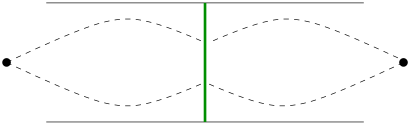
$R = \infty, R^* = \frac{\pi}{2} R_0$



$R = 0, R^* = 0$

$S_0^0 > 0$

$R = 0, R^* = 0$



$R = \infty, R^* = -\frac{\pi}{2} R_0$

$S_0^0 < 0$

Solutions $\Lambda < 0$, $f^2(t) + k > 0$

$$R_{,n} = \sigma \sqrt{f^2(t) + k - \frac{2\Lambda}{N(N-1)} R^2} = \sigma \sqrt{f^2(t) + k + \frac{R^2}{R_0^2}} ,$$

$$R = R_0 \sqrt{f^2(t) + k} \, \sinh \left(\frac{\sigma n}{R_0} + \varphi(t) \right) .$$

On the brane

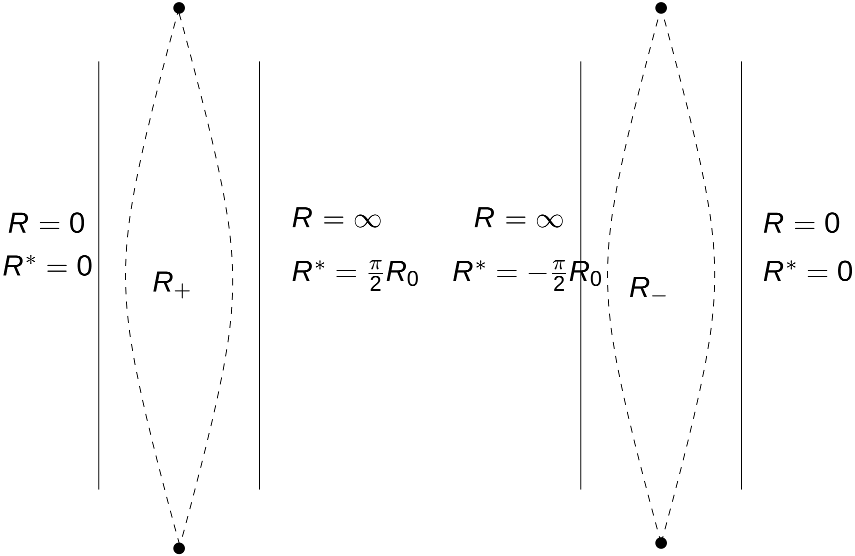
$$\frac{\sigma n}{R_0} \coth \varphi(t) = \frac{4\pi G}{N-1} S^0_0 ,$$

$$\frac{f^2(t) + k}{a^2} = \frac{a_\tau^2 + k}{a^2} = \frac{2\Lambda}{N(N-1)} + \left(\frac{4\pi G}{N-1} \right)^2 S^0_0{}^2 = \frac{1}{R_0^2 \sinh^2 \varphi(t)} .$$

"Heavy brane/shell"

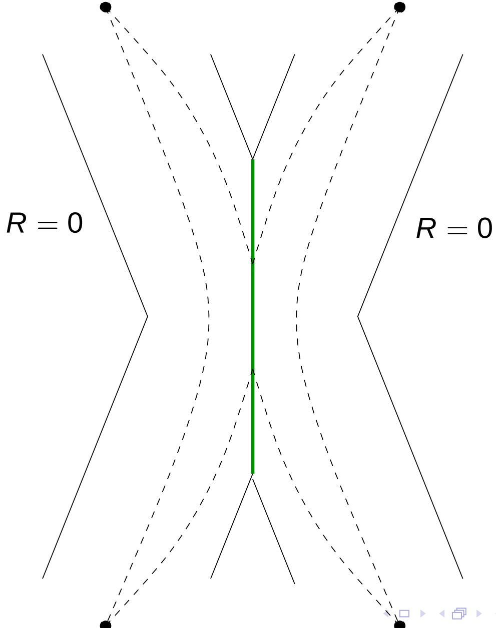
If $\left| S^0_0 \right| > \sqrt{\frac{(N-1)\,|\Lambda|}{2\pi G N}}$ evolution inside the brane is the same as for $\Lambda > 0$

Bulk: Carter-Penrose diagrams $\Lambda < 0$, $k = +1$, $f^2(t) + k > 0$

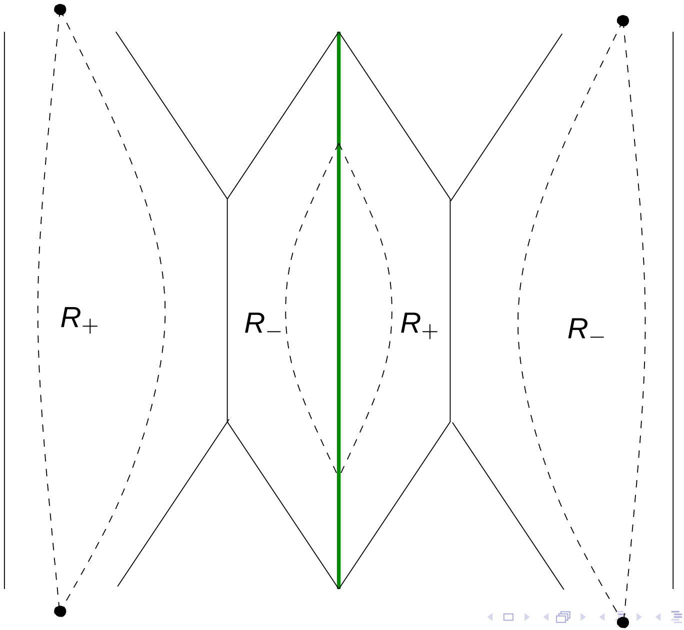


Unfolded AdS

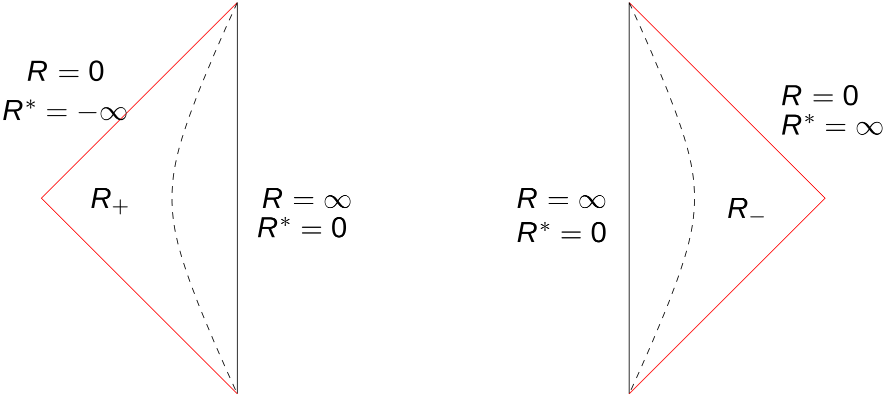
$\Lambda < 0, k = +1, S^0_0 > 0$



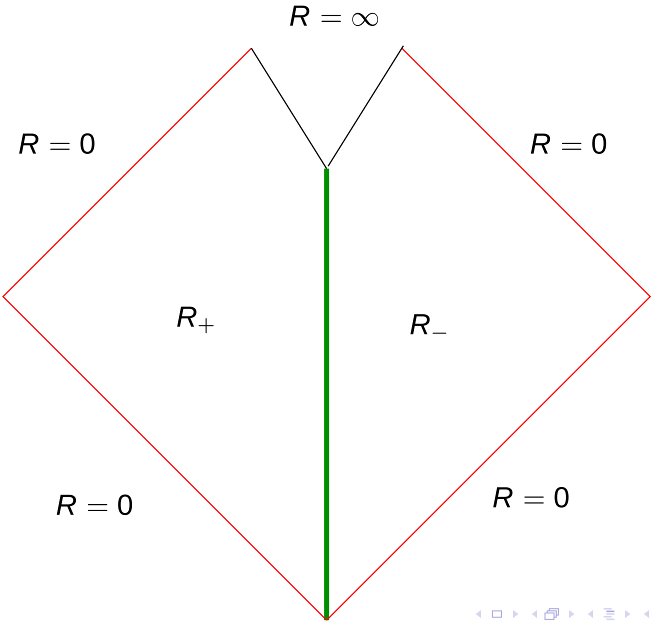
$\Lambda < 0, k = +1, S^0_0 < 0$



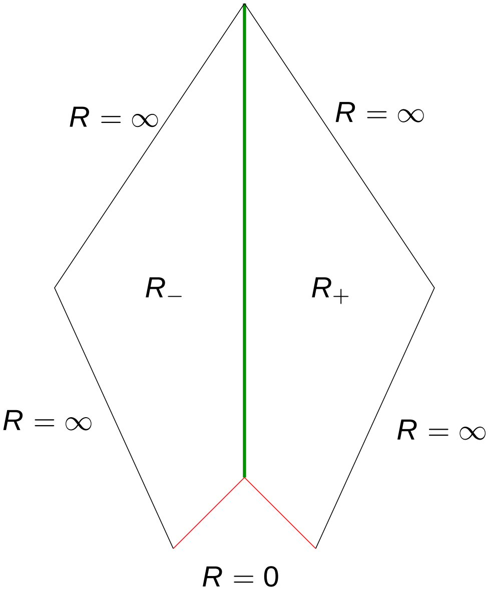
Bulk: Carter-Penrose diagrams $\Lambda < 0$, $k = 0$



$\Lambda < 0, k = 0, S_0^0 > 0$



$\Lambda < 0$, $k = 0$, $S_0^0 < 0$



Solutions $\Lambda < 0$, $k = -1$, $f^2(t) + k > 0$, $f^2(t) + k < 0$

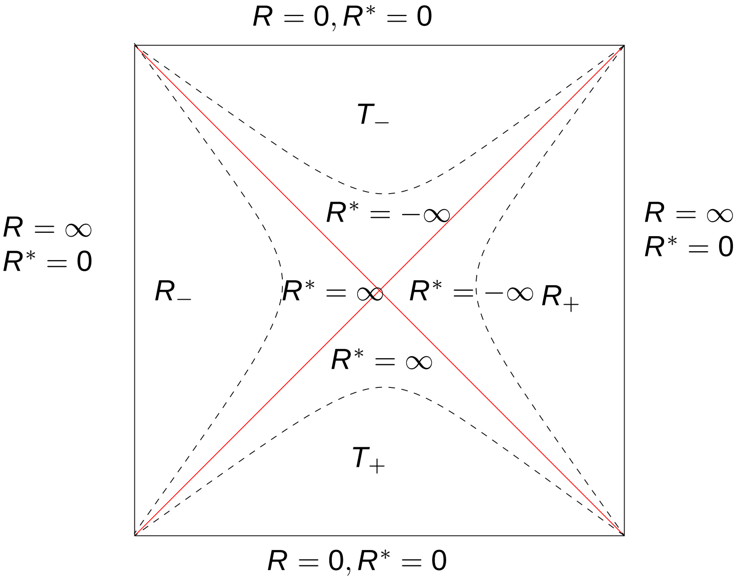
"Heavy shell" $\left|S_0^0\right| > \sqrt{\frac{(N-1)|\Lambda|}{2\pi GN}}$

$$R = R_0 \sinh\left(\frac{t}{R_0 \sinh\varphi_0}\right) \sinh\left(\frac{\sigma n}{R_0} + \varphi_0\right),$$

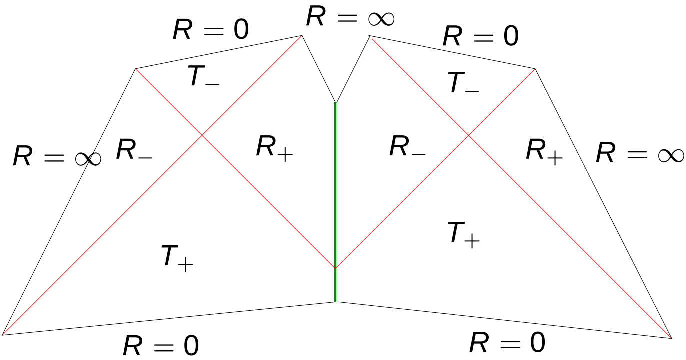
"Light shell" $\left|S_0^0\right| < \sqrt{\frac{(N-1)|\Lambda|}{2\pi GN}}$

$$\begin{aligned}
R &= R_0 \sin\left(\frac{t}{R_0 \cosh\varphi_0}\right) \cosh\left(\frac{\sigma n}{R_0} + \varphi_0\right), \\
R &= \pm R_0 \tanh\frac{R^\star}{R_0}, \qquad , 0 \leq R \leq R_0, \\
R &= R_0 \coth\frac{R^\star}{R_0}, \qquad R_0 \leq R < \infty.
\end{aligned}$$

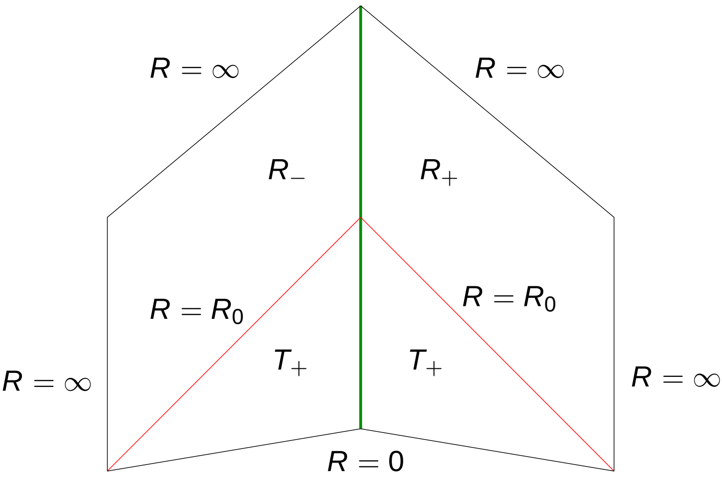
Bulk: Carter-Penrose diagrams $\Lambda < 0$, $k = -1$



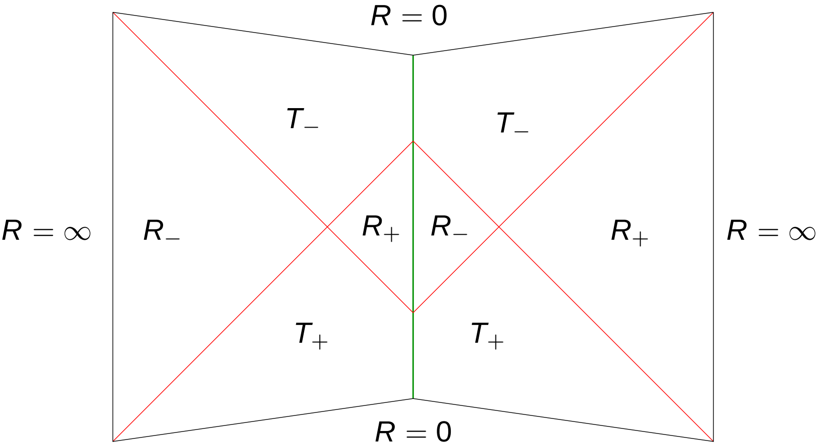
“Heavy brane” $\Lambda < 0, k = -1, S_0^0 > 0$



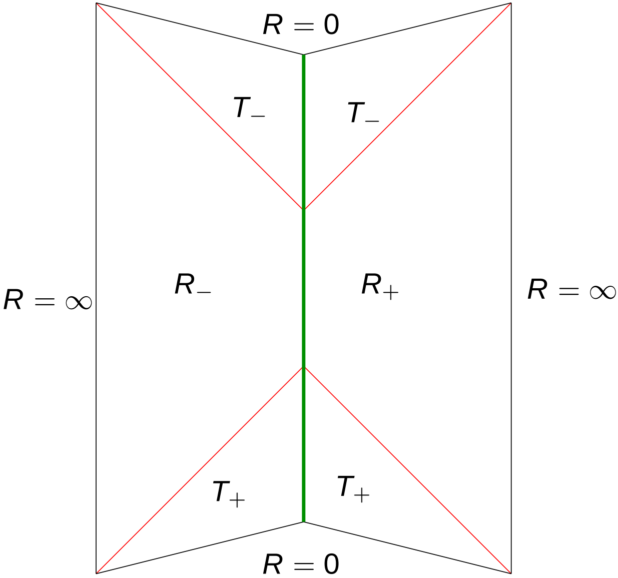
“Heavy brane” $\Lambda < 0$, $k = -1$, $S_0^0 < 0$



“Light brane” $\Lambda < 0$, $k = -1$, $S_0^0 > 0$



“Light brane” $\Lambda < 0$, $k = -1$, $S_0^0 < 0$



THANK YOU