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Revealing structure of quantum corrections in $N = 1$ supersymmetric theories using the Schwinger–Dyson equations

Quantum corrections in supersymmetric theories are investigated for a long time, for example in the papers

L.V.Avdeev, O.V.Tarasov, Phys.Lett. 112B, (1982),356;
A.Parkes, P.West, Phys.Lett. 138B, (1983), 99;
I.Jack, D.R.T.Jones, C.G.North, Phys.Lett B386, (1996), 138;
Nucl.Phys. B486 (1997), 479;
I.Jack, D.R.T.Jones, A.Pickering, Phys.Lett. B435, (1998), 61.

Most calculations were made with the dimensional reduction

W.Siegel, Phys.Lett. 84B, (1979), 193; 94B, (1980), 37.

With the dimensional reduction the β-function was calculated even in the four-loop approximation. It is in agreement with the exact NSVZ β-function

$$\beta(\alpha) = -\frac{\alpha^2\Big[3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)\Big]}{2\pi(1 - C_2\alpha/2\pi)}.$$

V.Novikov, M.A.Shifman, A.Vainstein, V.Zakharov, Nucl.Phys. B 229, (1983), 381; Phys.Lett. 166B, (1985), 329; M.Shifman, A.Vainshtein, Nucl.Phys. B 277, (1986), 456.

up to the redefinition of the coupling constant.

Calculations with the higher covariant derivatives

Interesting features can be revealed with the higher covariant derivative regularization.

A.A.Slavnov, Theor.Math.Phys. 23, (1975), 3; P.West, Nucl.Phys. B268, (1986), 113.

Usually integrals, obtained with the higher derivative regularization are very complicated. However for supersymmetric theories integrals, defining the β-function, can be easily calculated, because the integrands are total derivatives.

A.Soloshenko, K.S. hep-th/0304083; A.Pimenov, K.S., Theor.Math.Phys. 147, (2006), 687.

Example: N=1 supersymmetric Yang-Mills theory with matter in the massless case is described by the action

$$S = \frac{1}{2e^2}\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W_a C^{ab} W_b + \frac{1}{4}\int d^4x\,d^4\theta\,(\phi^*)^i (e^{2V})_i{}^j \phi_j +$$

$$+\Big(\frac{1}{6}\int d^4x\,d^2\theta\,\lambda^{ijk}\phi_i\phi_j\phi_k + \mbox{h.c.}\Big).$$

Two-loop β-function for a general renormalizable $N=1$ supersymmetric Yang-Mills theory

Two-loop calculation gives the following result:

$$\beta(\alpha) = -\frac{3\alpha^2}{2\pi}C_2 + \alpha^2 T(R) I_0 + \alpha^3 C_2^2 I_1 + \frac{\alpha^3}{r} C(R)_i{}^j C(R)_j{}^i I_2 + $$
$$+\alpha^3 T(R) C_2 I_3 + \alpha^2 C(R)_i{}^j \frac{\lambda^*_{jkl}\lambda^{ikl}}{4\pi r} I_4 + \ldots,$$

where we do not write the integral for the one-loop ghost contribution and the integrals I_0–I_4 are given below, and the following notation is used:

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA}.$$

Taking into account Pauli–Villars contributions,

$$I_i = I_i(0) - \sum_I I_i(M_I), \qquad i = 0, 2, 3$$

where I_i are given by

Factorization of integrands into total derivatives

$$I_0(M) = 8\pi \int \frac{d^4q}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{q^2} \frac{d}{dq^2} \Bigg\{ \frac{1}{2} \ln \Big(q^2 (1 + q^{2m}/\Lambda^{2m})^2 + M^2 \Big) +$$

$$+ \frac{M^2}{2(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)} - \frac{m q^{2m}/\Lambda^{2m} q^2 (1 + q^{2m}/\Lambda^{2m})}{q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2} \Bigg\};$$

$$I_1 = 96\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{k^2} \frac{d}{dk^2} \Bigg\{ \frac{1}{q^2(q+k)^2(1+q^{2n}/\Lambda^{2n})} \times$$

$$\times \frac{1}{(1+(q+k)^{2n}/\Lambda^{2n})} \left(\frac{n+1}{(1+k^{2n}/\Lambda^{2n})} - \frac{n}{(1+k^{2n}/\Lambda^{2n})^2} \right) \Bigg\};$$

$$I_2(M) = -64\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{q^2} \frac{d}{dq^2} \Bigg\{ \frac{q^2}{k^2(1+k^{2n}/\Lambda^{2n})} \times$$

$$\times \frac{(1+(q+k)^{2m}/\Lambda^{2m})}{((q+k)^2(1+(q+k)^{2m}/\Lambda^{2m}) + M^2)} \Bigg[\frac{q^2(1+q^{2m}/\Lambda^{2m})^3}{(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)^2} +$$

$$+ \frac{m q^{2m}/\Lambda^{2m}}{q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2} - \frac{2m q^{2m}/\Lambda^{2m} M^2}{(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)^2} \Bigg] \Bigg\};$$

Factorization of integrands into total derivatives

$$I_3(M) = 16\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln\Lambda} \left\{ \frac{\partial}{\partial q_\alpha} \left[\frac{k_\alpha (1+q^{2m}/\Lambda^{2m})}{(q^2(1+q^{2m}/\Lambda^{2m})^2 + M^2)} \times \right.\right.$$

$$\times \frac{1}{(k+q)^2(1+(q+k)^{2n}/\Lambda^{2n})} \Big(- \frac{(1+k^{2m}/\Lambda^{2m})^3}{(k^2(1+k^{2m}/\Lambda^{2m})^2+M^2)^2} +$$

$$\left. + \frac{mk^{2m}/\Lambda^{2m}}{k^2(1+k^{2m}/\Lambda^{2m})^2+M^2} - \frac{2mk^{2m}/\Lambda^{2m} M^2}{(k^2(1+k^{2m}/\Lambda^{2m})^2+M^2)^2} \Big) \right] -$$

$$- \frac{1}{k^2} \frac{d}{dk^2} \left[\frac{2(1+q^{2m}/\Lambda^{2m})(1+(q+k)^{2m}/\Lambda^{2m})}{(q^2(1+q^{2m}/\Lambda^{2m})^2+M^2)\left((q+k)^2(1+(q+k)^{2m}/\Lambda^{2m})^2+M^2\right)} \times \right.$$

$$\left.\left. \times \Big(\frac{1}{(1+k^{2n}/\Lambda^{2n})} + \frac{nk^{2n}/\Lambda^{2n}}{(1+k^{2n}/\Lambda^{2n})^2} \Big) \right] \right\};$$

$$I_4 = 64\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{1}{q^2} \frac{d}{dq^2} \left[\frac{1}{k^2(q+k)^2(1+k^{2m}/\Lambda^{2m})} \times \right.$$

$$\left. \times \frac{1}{(1+(q+k)^{2m}/\Lambda^{2m})} \Big(\frac{1}{(1+q^{2m}/\Lambda^{2m})} + \frac{mq^{2m}/\Lambda^{2m}}{(1+q^{2m}/\Lambda^{2m})^2} \Big) \right].$$

Two-loop β-function for a general renormalizable $N=1$ supersymmetric Yang-Mills theory

The integrals can be calculated using the identity

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{d}{dk^2} f(k^2) = \frac{1}{16\pi^2} \Big(f(k^2=\infty) - f(k^2=0) \Big).$$

The result for the two-loop Gell-Mann–Low function is given by

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha^3}{(2\pi)^2}\Big(-3C_2^2 + T(R)C_2 + \\ + \frac{2}{r} C(R)_i{}^j C(R)_j{}^i \Big) - \frac{\alpha^2 C(R)_i{}^j \lambda^*_{jkl}\lambda^{ikl}}{8\pi^3 r} + \ldots.$$

Comparing this with the one-loop anomalous dimension gives the NSVZ result. The result also agrees with the DRED calculations.

Three-loop calculation for SQED

The notation is

$$\Gamma^{(2)} = \int \frac{d^4p}{(2\pi)^4}\, d^4\theta\,\Big(- \frac{1}{16\pi}\mathbf{V}(-p)\,\partial^2\Pi_{1/2}\mathbf{V}(p)\, d^{-1}(\alpha,\mu/p) + \\ + \frac{1}{4}(\phi^*)^i(-p,\theta)\,\phi_j(p,\theta)\,(ZG)_i{}^j(\alpha,\mu/p)\Big). \qquad (1)$$

The main result: (It was obtained as the equality of some well defined integrals)

$$\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} = -\frac{d}{d\ln\Lambda}\alpha_0^{-1}(\alpha,\mu/\Lambda) = \\ = \frac{1}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big) = \frac{1}{\pi} + \frac{1}{\pi}\frac{d}{d\ln\Lambda}\Big(\ln ZG(\alpha,\mu/q) - \\ - \ln Z(\alpha,\Lambda/\mu)\Big)\Big|_{q=0} = \frac{1}{\pi}\Big(1-\gamma\Big(\alpha_0(\alpha,\Lambda/\mu)\Big)\Big).$$

Therefore, with higher derivatives it is not necessary to redefine the coupling constant in order to obtain the agreement with the NSVZ β-function. The reason is that the integrands are again total derivatives.

Possible explanation

One can try to explain the factorization of integrands into total derivatives substituting solutions of the Slavnov–Taylor identities into the Schwinger–Dyson equations.

Here we consider only a contribution of the matter superfields: (omitting the regularization in order to avoid large expressions)

$$\frac{\delta\Gamma}{\delta\mathbf{V}_y^B\delta\mathbf{V}_x^A} = \frac{e}{4}\frac{\delta}{\delta\mathbf{V}_y^B}\Big\langle(\phi_x^*)^i(T^A)_i{}^j(e^{2V_x'}\phi_x)_j + \text{h.c.}\Big\rangle + \text{gauge contribution.}$$

Then we introduce the notation

$$(\mathbf{U}_0)^i \equiv \sum_{n=1}^{\infty}\lambda^{ijk}\Big[\frac{D^2}{2\partial^2}\Big(\phi_j\phi_k\Big) + 2\Big(\frac{D^2}{2\partial^2}\phi_j\Big)\phi_k\Big];$$

$$(\mathbf{U}_1)^i \equiv \sum_{n=1}^{\infty}\lambda^{ijk}\phi_j\phi_k. \qquad (2)$$

We also need auxiliary sources for these values:

$$S_{Source} = \int d^4x\, d^2\theta\, (\varphi)_j (\mathbf{U}_1)^j + \frac{1}{4}\int d^8x\, (\phi_0^*)^i (e^{2V'}\phi + \mathbf{U}_0^*)_i + \text{h.c.}$$

There is a very important identity

$$-\frac{D^2}{2}\frac{\delta\Gamma}{\delta(\phi_0^*)^i} = -\frac{D^2}{8}\Big\langle (e^{2V'}\phi + \mathbf{U}_0^*)_i \Big\rangle = \frac{\delta\Gamma}{\delta(\phi^*)^i},$$

which allows to relate usual Green functions, and the functions, containing derivatives w.r.t. auxiliary sources.

Then using the identity

$$\int d^8x\, \mathbf{V}^A (\phi^* e^{2V'})^i (T^A)_i{}^j \phi_j = \int d^8x\, \Big\{ \Big(8\frac{\delta \mathbf{S}}{\delta\phi_i}\mathbf{V}^A - (\phi^* e^{2V'} + $$

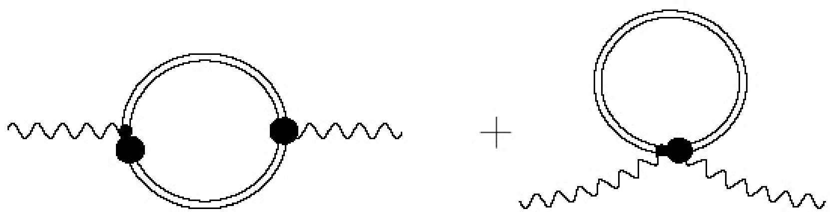
$$+\mathbf{U}_0)^i \bar{D}^2 \mathbf{V}^A - 2\bar{D}^a (\phi^* e^{2V'} + \mathbf{U}_0)^i \bar{D}_a \mathbf{V}^A \Big) (T^A)_i{}^j \frac{D^2}{16\partial^2}\phi_j -$$

$$-\frac{D^2}{2\partial^2}(\mathbf{U}_1)^i (T^A)_i{}^j \Big(\bar{D}^2\mathbf{V}^A \frac{D^2}{16\partial^2}\phi_j + 2\bar{D}^a \mathbf{V}^A \frac{\bar{D}_a D^2}{16\partial^2}\phi_j \Big) \Big\}. \qquad (3)$$

the considered contribution to the two-point function can be written as a sum of some effective diagrams.

Calculating matter contribution by Schwinger-Dyson equations and Slavnov-Taylor identities

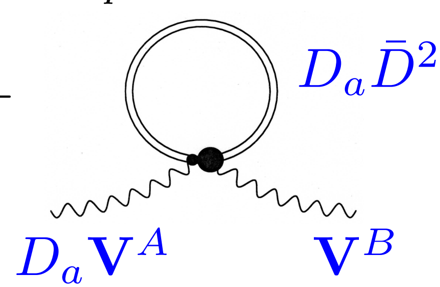
Graphically



Vertexes here contain derivatives w.r.t. auxiliary sources ϕ_0 and φ. They are restricted by the Slavnov–Taylor identities. Substituting solution of these identities we obtain (in the massless case for simplicity)

$$\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\lambda_0,\Lambda/p)-\alpha_0^{-1}\Big)\Big|_{p=0} = -2\pi T(R)\frac{d}{d\ln\Lambda}\int\frac{d^4q}{(2\pi)^4}\times$$

$$\times\frac{1}{q^2}\frac{d}{dq^2}\Big(\ln(q^2G^2)+2K_\varphi\Big)+$$



The function K_φ is defined by

$$\frac{\delta^2\Gamma}{\delta(\mathbf{j}_y)^j\delta(\varphi_w)_i} \equiv -\frac{1}{2}(K_\varphi)_j{}^i \bar{D}_y^2\delta^8_{yw}. \qquad (4)$$

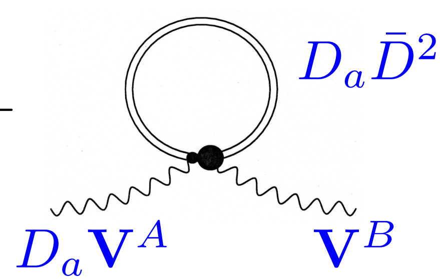
For a theory with cubic superpotential this function is nontrivial in the two-loop approximation.

Factorization of some other values into total derivatives

The explanation, given above, is not complete, because some explicit calculations show that the diagram, which is contributed by the transversal part of the Green function, in the lowest loops is also an integral of a total derivative.

Therefore, it is necessary to explain, why this contribution disappear. More exactly, it is necessary to explain why in the in lowest loops

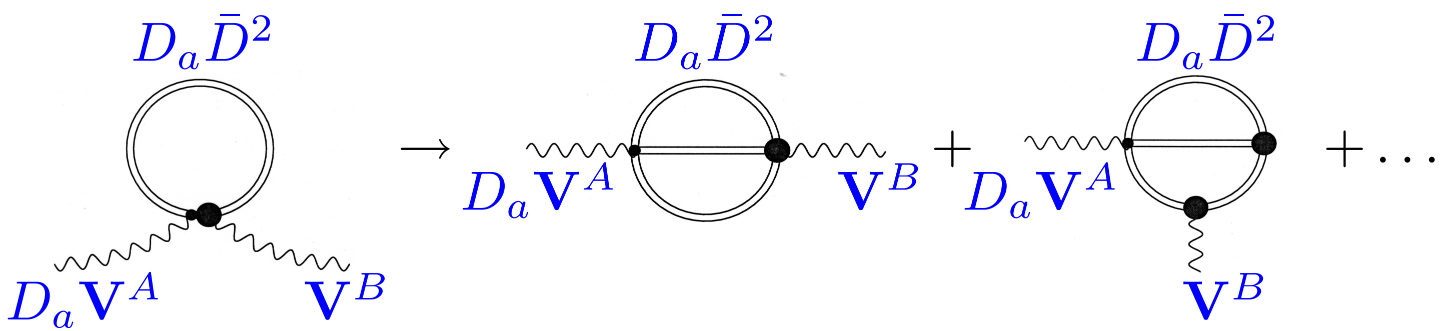
$$\frac{1}{4\pi}T(R)\frac{dK_\varphi}{d\ln\Lambda}\Big|_{q=0} +$$



$$= 0$$

Toward the explanation

The diagram, which was not calculated, contains derivatives with respect to the auxiliary sources ϕ_0. We consider only a part, corresponding to the cubic superpotential. It can be rewritten as a sum of two-loop effective diagrams:



These diagrams are calculated by the same method as above. The result is the following: The "bad" term with the function K_φ is completely canceled. However, again there is a nontrivial contribution of the transversal part (staring at least from the three-loop approximation). Possibly, it can be zero, but so far there are no calculations, confirming this.

Conclusion and open questions

- ✓ With the higher derivative regularization integrals, defining the β-function, at least in the lowest loops can be easily taken, because the integrands are total derivatives. The result is in agreement with the NSVZ β-function.
- ✓ The explanation can be possibly made by substituting solutions of the Slavnov–Taylor identities into the Schwinger–Dyson equations. The main problem is whether the factorization of integrands into total derivatives takes place only in the lowest loops or it holds exactly to all loops.
- ✓ Higher loops in the Schwinger–Dyson equations are essential and can possibly lead to explanation of some so far mysterious cancelations.