# Energy of unstable states at long times

K. Urbanowski[1], J. Piskorski[2]
University of Zielona Góra, Institute of Physics,
ul. Prof. Z. Szafrana 4a, 65–516 Zielona Góra, Poland.

---

[1]e–mail: K.Urbanowski@proton.if.uz.zgora.pl; K.Urbanowski@if.uz.zgora.pl
[2]e–mail: J.Piskorski@proton.if.uz.zgora.pl

# 1. Introduction

Within the quantum theory the state vector at time $t$, $|\Phi(t)\rangle$, for the physical system under consideration which initially (at $t = t_0 = 0$) was in the state $|\Phi\rangle$ can be found by solving the Schödinger equation

$$i\hbar \frac{\partial}{\partial t}|\Phi(t)\rangle = H|\Phi(t)\rangle, \quad |\Phi(0)\rangle = |\Phi\rangle, \quad (1)$$

where $|\Phi(t)\rangle, |\Phi\rangle \in \mathcal{H}$, $\mathcal{H}$ is the Hilbert space of states of the considered system, $\| |\Phi(t)\rangle \| = \| |\Phi\rangle \| = 1$ and $H$ denotes the total selfadjoint Hamiltonian for the system. If one considers an unstable state $|\Phi\rangle \equiv |\phi\rangle$ of the system then using the solution $|\phi(t)\rangle$ of Eq. (1) for the initial condition $|\phi(0)\rangle = |\phi\rangle$ one can determine the decay law, $\mathcal{P}_\phi(t)$ of this state decaying in vacuum

$$\mathcal{P}_\phi(t) = |a(t)|^2, \quad (2)$$

where $a(t)$ is probability amplitude of finding the system at the time $t$ in the initial state $|\phi\rangle$ prepared at time $t_0 = 0$,

$$a(t) = \langle\phi|\phi(t)\rangle. \quad (3)$$

We have

$$a(0) = 1. \tag{4}$$

From basic principles of quantum theory it is known that the amplitude $a(t)$, and thus the decay law $\mathcal{P}_\phi(t)$ of the unstable state $|\phi\rangle$, are completely determined by the density of the energy distribution $\omega(\mathcal{E})$ for the system in this state
( S. Krylov, V. A. Fock, Zh. Eksp. Teor. Fiz. **17**, (1947), 93.),

$$a(t) = \int_{Spec.(H)} \omega(\mathcal{E}) \, e^{-\frac{i}{\hbar} \mathcal{E} \, t} \, d\mathcal{E}. \tag{5}$$

where $\omega(\mathcal{E}) \geq 0$.
Note that (5) and (4) mean that there must be

$$a(0) = \int_{Spec.(H)} \omega(\mathcal{E}) \, d\mathcal{E} = 1. \tag{6}$$

From the last property and from the Riemann–Lebesgue Lemma it follows that the amplitude $a(t)$, being the Fourier transform of $\omega(\mathcal{E})$ (see (5)), must tend to zero as $t \to \infty$.

In L. A. Khalfin paper, Zh. Eksp. Teor. Fiz. **33**, (1957), 1371; [Sov. Phys. – JETP **6**, (1958), 1053] , assuming that the spect–rum of $H$ must be bounded from below, $(Spec.(H) > -\infty)$, and using the Paley–Wiener Theorem it was proved that in the case of unstable states there must be

$$|a(t)| \geq A\,e^{-b\,t^q}, \tag{7}$$

for $|t| \to \infty$. Here $A > 0$, $b > 0$ and $0 < q < 1$.

This means that the decay law $\mathcal{P}_\phi(t)$ of unstable states decaying in the vacuum, (2), can not be described by an exponential function of time $t$ if time $t$ is suitably long, $t \to \infty$, and that for these lengths of time $\mathcal{P}_\phi(t)$ tends to zero as $t \to \infty$ more slowly than any exponential function of $t$. The analysis of the models of the decay processes shows that $\mathcal{P}_\phi(t) \simeq \exp(-\frac{\gamma_\phi^0 t}{\hbar})$, (where $\gamma_\phi^0$ is the decay rate of the state $|\phi\rangle$), to an very high accuracy for a wide time range $t$: From $t$ suitably later than some $T_0 \simeq t_0 = 0$ but $T_0 > t_0$ up to $t \gg \tau_\phi = \frac{\hbar}{\gamma_\phi^0}$ and smaller than $t = t_{as}$, where $t_{as}$ denotes the time $t$ for which the nonexponential deviations of $a(t)$ begin to dominate.

From this analysis it follows that in the general case the decay law $\mathcal{P}_\phi(t)$ takes the inverse power–like form $t^{-\lambda}$, (where $\lambda > 0$), for suitably large $t \geq t_{as} \gg \tau_\phi$. This effect is in agreement with the general result (7). Effects of this type are sometimes called the "Khalfin effect"

The problem how to detect possible deviations from the exponential form of $\mathcal{P}_\phi(t)$ at the long time region has been attracting the attention of physicists since the first theoretical predictions of such an effect. Many tests of the decay law performed some time ago did not indicate any deviations from the exponential form of $\mathcal{P}_\phi(t)$ at the long time region. Nevertheless conditions leading to the nonexponetial behavior of the amplitude $a(t)$ at long times were studied theoretically. Conclusions following from these studies were applied successfully in experiment described in the Rothe paper

( C. Rothe, S. I. Hintschich and A. P. Monkman, Phys. Rev. Lett. **96**,(2006), 163601. ), where the experimental evidence of the exponential decay law at long times was reported. This result gives rise to another problem which now becomes important:

If (and how) deviations from the exponential decay law at long times affect the energy of the unstable state and its decay rate at this time region.

Note that in fact the amplitude $a(t)$ contains information about the decay law $\mathcal{P}_\phi(t)$ of the state $|\phi\rangle$, that is about the decay rate $\gamma_\phi^0$ of this state, as well as the energy $\mathcal{E}_\phi^0$ of the system in this state. This information can be extracted from $a(t)$. Indeed if $|\phi\rangle$ is an unstable (a quasi–stationary) state then

$$a(t) \cong e^{-\frac{i}{\hbar}\left(\mathcal{E}_\phi^0 - \frac{i}{2}\gamma_\phi^0\right) t}. \tag{8}$$

So, there is

$$\mathcal{E}_\phi^0 - \frac{i}{2}\gamma_\phi^0 \equiv i\hbar \, \frac{\partial a(t)}{\partial t} \, \frac{1}{a(t)}, \tag{9}$$

in the case of quasi–stationary states.

The standard interpretation and understanding of the quantum theory and the related construction of our measuring devices are such that detecting the energy $\mathcal{E}_\phi^0$ and decay rate $\gamma_\phi^0$ one is sure that the amplitude $a(t)$ has the form (8) and thus that the relation (9) occurs. Taking the above into account one can define the "effective Hamiltonian", $h_\phi$, for the one–dimensional subspace of states $\mathcal{H}_{||}$ spanned by the normalized vector $|\phi\rangle$ as follows

$$h_\phi \stackrel{\text{def}}{=} i\hbar \, \frac{\partial a(t)}{\partial t} \, \frac{1}{a(t)}. \qquad (10)$$

In general, $h_\phi$ can depend on time $t$, $h_\phi \equiv h_\phi(t)$. One meets this effective Hamiltonian when one starts with the Schrödinger Equation (1) for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for the distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$. In the case of one–dimensional $\mathcal{H}_{||}$ this rigorous Schrödinger–like evolution equation has the following form for the initial condition $a(0) = 1$,

$$i\hbar \frac{\partial a(t)}{\partial t} = h_\phi(t)\, a(t). \qquad (11)$$

Relations (10) and (11) establish a direct connection between the amplitude $a(t)$ for the state $|\phi\rangle$ and the exact effective Hamiltonian $h_\phi(t)$ governing the time evolution in the one–dimensional subspace $\mathcal{H}_\parallel \ni |\phi\rangle$. Thus the use of the evolution equation (11) or the relation (10) is one of the most effective tools for the accurate analysis of the early– as well as the long–time properties of the energy and decay rate of a given quasi–stationary state $|\phi(t)\rangle$. So let us assume that we know the amplitude $a(t)$. Then starting with this $a(t)$ and using the expression (10) one can calculate the effective Hamiltonian $h_\phi(t)$ in a general case for every $t$. Thus, one finds the following expressions for the energy and the decay rate of the system in the state $|\phi\rangle$ under considerations, to be more precise for the instantaneous energy and the instantaneous decay rate,

$$\mathcal{E}_\phi \equiv \mathcal{E}_\phi(t) = \Re\left(h_\phi(t)\right), \qquad (12)$$
$$\gamma_\phi \equiv \gamma_\phi(t) = -2\,\Im\left(h_\phi(t)\right), \qquad (13)$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of $z$
respectively.

The deviations of the decay law $\mathcal{P}_\phi(t)$ from the exponential form
can be described equivalently using time-dependent decay rate
(13). In terms of such $\gamma_\phi(t)$ the Khalfin observation that $\mathcal{P}_\phi(t)$
must tend to zero as $t \to \infty$ more slowly than any exponential
function means that $\gamma_\phi(t) \ll \gamma_\phi^0$ for $t \gg t_{as}$ and
$\lim_{t\to\infty} \gamma_\phi(t) = 0$.

Using (10) and (12), (13) one can find that

$$\mathcal{E}_\phi(0) = \langle\phi|H|\phi\rangle, \tag{14}$$

$$\mathcal{E}_\phi(t \sim \tau_\phi) \simeq \mathcal{E}_\phi^0 \neq \mathcal{E}_\phi(0), \tag{15}$$

$$\gamma_\phi(0) = 0, \tag{16}$$

$$\gamma_\phi(t \sim \tau_\phi) \simeq \gamma_\phi^0. \tag{17}$$

The aim of this talk is to discuss the long time behaviour of $\mathcal{E}_\phi(t)$ using $a(t)$ calculated for the given density $\omega(\mathcal{E})$. We show that $\mathcal{E}_\phi(t) \to \mathcal{E}_{min} > -\infty$ as $t \to \infty$ for the model considered and that a wide class of models has similar long time properties: $\mathcal{E}_\phi(t)|_{t \to \infty} \neq \mathcal{E}_\phi^0$. It seems that in contrast to the standard Khalfin effect in the case of the quasi–stationary states belonging to the same class as excited atomic levels, this long time properties of the instantaneous energy $\mathcal{E}_\phi(t)$ have a chance to be detected, eg., by analyzing properties of the high energy cosmic rays or spectra of very distant stars.

## 2. The model

Let us assume that
$Spec.(H) = [\mathcal{E}_{min}, \infty)$,
(where, $\mathcal{E}_{min} > -\infty$), and let us choose $\omega(\mathcal{E})$ as follows

$$\omega(\mathcal{E}) \equiv \omega_{BW}(\mathcal{E}) = \frac{N}{2\pi} \, \Theta(\mathcal{E} - \mathcal{E}_{min}) \, \frac{\gamma_\phi^0}{(\mathcal{E} - \mathcal{E}_\phi^0)^2 + (\frac{\gamma_\phi^0}{2})^2}, \quad (18)$$

where $N$ is a normalization constant and

$$\Theta(\mathcal{E}) = \begin{cases} 1 & \text{for } \mathcal{E} \geq 0, \\ 0 & \text{for } \mathcal{E} < 0, \end{cases}$$

For such $\omega(\mathcal{E})$ using (5) one has

$$a(t) = \frac{N}{2\pi} \int_{\mathcal{E}_{min}}^{\infty} \frac{\gamma_\phi^0}{(\mathcal{E} - \mathcal{E}_\phi^0)^2 + (\frac{\gamma_\phi^0}{2})^2} \, e^{-\frac{i}{\hbar}\mathcal{E}t} \, d\mathcal{E}, \qquad (19)$$

where

$$\frac{1}{N} = \frac{1}{2\pi} \int_{\mathcal{E}_{min}}^{\infty} \frac{\gamma_\phi^0}{(\mathcal{E} - \mathcal{E}_\phi^0)^2 + (\frac{\gamma_\phi^0}{2})^2} \, d\mathcal{E}. \qquad (20)$$

Formula (19) leads to the result

$$
\begin{aligned}
a(t) \;=\; & N\, e^{-\frac{i}{\hbar}(\mathcal{E}_\phi^0 - i\frac{\gamma_\phi^0}{2})t}\Big\{ 1 - \frac{i}{2\pi}\times \\
& \times \Big[ e^{\frac{\gamma_\phi^0 t}{\hbar}}\, E_1\Big( -\frac{i}{\hbar}(\mathcal{E}_\phi^0 - \mathcal{E}_{min} + \frac{i}{2}\gamma_\phi^0)t \Big) \\
& - E_1\Big( -\frac{i}{\hbar}(\mathcal{E}_\phi^0 - \mathcal{E}_{min} - \frac{i}{2}\gamma_\phi^0)t \Big) \Big] \Big\}, \qquad (21)
\end{aligned}
$$

where $E_1(x)$ denotes the integral–exponential function.

In general one has

$$a(t) \equiv a_{exp}(t) + a_{non}(t), \qquad (22)$$

where

$$a_{exp}(t) = N\, e^{-\frac{i}{\hbar}(\mathcal{E}_\phi^0 - i\frac{\gamma_\phi^0}{2})t}, \qquad a_{non}(t) = a(t) - a_{exp}(t).$$

Making use of the asymptotic expansion of $E_1(x)$

$$E_1(z)\big|_{|z|\to\infty} \;\sim\; \frac{e^{-z}}{z}\left(1 - \frac{1}{z} + \frac{2}{z^2} - \ldots\right), \qquad (23)$$

where $|\arg z| < \frac{3}{2}\pi$, one finds

$$a(t)|_{t \to \infty} \simeq N e^{-\frac{i}{\hbar} h_\phi^0 t}$$
$$+ \frac{N}{2\pi} e^{-\frac{i}{\hbar} \mathcal{E}_{min} t} \left\{ (-i) \frac{\gamma_\phi^0}{|h_\phi^0 - \mathcal{E}_{min}|^2} \frac{\hbar}{t} \right.$$
$$\left. -2 \frac{(\mathcal{E}_\phi^0 - \mathcal{E}_{min}) \gamma_\phi^0}{|h_\phi^0 - \mathcal{E}_{min}|^4} \left( \frac{\hbar}{t} \right)^2 + \dots \right\} \quad (24)$$

and

$$
\begin{aligned}
h_\phi(t)\big|_{t\to\infty} &= i\hbar \frac{\partial a(t)}{\partial t}\frac{1}{a(t)}\bigg|_{t\to\infty} \\
&\simeq \mathcal{E}_{min} - i\frac{\hbar}{t} - 2\frac{\mathcal{E}_\phi^0 - \mathcal{E}_{min}}{|h_\phi^0 - \mathcal{E}_{min}|^2}\left(\frac{\hbar}{t}\right)^2 + \ (25)
\end{aligned}
$$

for the considered case (18) of $\omega_{BW}(\mathcal{E})$ (for details see:

).

From (25) it follows that

$$\Re\left(h_\phi(t)\big|_{t\to\infty}\right) \stackrel{\text{def}}{=} \mathcal{E}_\phi^\infty(t)$$

$$\simeq \mathcal{E}_{min} - 2\frac{\mathcal{E}_\phi^0 - \mathcal{E}_{min}}{|\,h_\phi^0 - \mathcal{E}_{min}\,|^2}\left(\frac{\hbar}{t}\right)^2$$

$$\underset{t\to\infty}{\longrightarrow} \mathcal{E}_{min}, \qquad (26)$$

where $\mathcal{E}_\phi^\infty(t) = \mathcal{E}_\phi(t)\big|_{t\to\infty}$, and

$$\Im\left(h_\phi(t)\big|_{t\to\infty}\right) \simeq -\frac{\hbar}{t} \underset{t\to\infty}{\longrightarrow} 0. \qquad (27)$$

The property (26) means that

$$\Re\left(h_\phi(t)|_{t\to\infty}\right) \equiv \mathcal{E}_\phi^\infty(t) < \mathcal{E}_\phi^0. \qquad (28)$$

For different states $|\phi\rangle = |j\rangle$, $(j = 1, 2, 3, \ldots)$ one has

$$\Im\left(h_1(t)|_{t\to\infty}\right) = \Im\left(h_2(t)|_{t\to\infty}\right), \qquad (29)$$

whereas in general $\gamma_1^0 \neq \gamma_2^0$.

Note that from (24) one obtains

$$|a(t)|_{t \to \infty}|^2 \simeq N^2 e^{-\frac{\gamma_\phi^0}{\hbar} t} + \frac{N^2}{4\pi^2} \frac{(\gamma_\phi^0)^2}{|h_\phi^0 - \mathcal{E}_{min}|^4} \frac{\hbar^2}{t^2} + \dots \ . \quad (30)$$

Relations (24) — (29) become important for times $t > t_{as}$, where $t_{as}$ denotes the time $t$ at which contributions to $|a(t)|_{t \to \infty}|^2$ from the first exponential component in (30) and from the second component proportional to $\frac{1}{t^2}$ are comparable, that is (see (22)),

$$|a_{exp}(t)|^2 \simeq |a_{non}(t)|^2 \quad (31)$$

for $t \to \infty$. So $t_{as}$ can be be found by considering the following relation

$$e^{-\frac{\gamma_\phi^0}{\hbar} t} \sim \frac{1}{4\pi^2} \frac{(\gamma_\phi^0)^2}{|h_\phi^0 - \mathcal{E}_{min}|^4} \frac{\hbar^2}{t^2}. \qquad (32)$$

Assuming that the right hand side is equal to the left hand side in the above relation one gets a transcendental equation. Exact solutions of such an equation can be expressed by means of the Lambert $W$ function.

An asymptotic solution of the equation obtained from the relation (32) is relatively easy to find. The very approximate asymptotic solution, $t_{as}$, of this equation for $(\frac{\mathcal{E}_\phi}{\gamma_\phi^0}) > 10^2$ (in general for $(\frac{\mathcal{E}_\phi}{\gamma_\phi^0}) \to \infty$) has the form

$$
\begin{aligned}
\frac{\gamma_\phi^0 t_{as}}{\hbar} \quad \simeq \quad & 8,28 + 4 \ln\left(\frac{\mathcal{E}_\phi^0 - \mathcal{E}_{min}}{\gamma_\phi^0}\right) \\
& + 2 \ln\left[8,28 + 4 \ln\left(\frac{\mathcal{E}_\phi^0 - \mathcal{E}_{min}}{\gamma_\phi^0}\right)\right] + \dots \quad . \quad (33)
\end{aligned}
$$

# 3. Some generalizations

To complete the analysis performed in the previous Section let us consider a more general case of $\omega(\mathcal{E})$ and $a(t)$. For a start let us consider relatively simple case when $\lim_{\mathcal{E} \to \mathcal{E}_{min}+} \omega(\mathcal{E}) \stackrel{\text{def}}{=} \omega_0 > 0$. Let derivatives $\omega^{(k)}(\mathcal{E})$, $(k = 0, 1, 2, \ldots, n)$, be continuous in $[\mathcal{E}_{min}, \infty)$, (that is let for $\mathcal{E} > \mathcal{E}_{min}$ all $\omega^{(k)}(\mathcal{E})$ be continuous and all the limits $\lim_{\mathcal{E} \to \mathcal{E}_{min}+} \omega^{(k)}(\mathcal{E})$ exist) and let all these $\omega^{(k)}(\mathcal{E})$ be absolutely integrable functions then (see: )

$$a(t) \underset{t \to \infty}{\sim} -\frac{i\hbar}{t} e^{-\frac{i}{\hbar}\mathcal{E}_{min} t} \sum_{k=0}^{n-1} (-1)^k \left(\frac{i\hbar}{t}\right)^k \omega_0^{(k)}, \quad (34)$$

where $\omega_0^{(k)} \stackrel{\text{def}}{=} \lim_{\mathcal{E} \to \mathcal{E}_{min}+} \omega^{(k)}(\mathcal{E})$.

Let us now consider a more complicated form of the density $\omega(\mathcal{E})$.
Namely let $\omega(\mathcal{E})$ be of the form

$$\omega(\mathcal{E}) = (\mathcal{E} - \mathcal{E}_{min})^\lambda \, \eta(\mathcal{E}) \; \in \; L_1(-\infty, \infty), \qquad (35)$$

where $0 < \lambda < 1$ and it is assumed that $\eta^{(k)}(\mathcal{E})$, $(k = 0, 1, \ldots, n)$,
exist and they are continuous in $[\mathcal{E}_{min}, \infty)$, and limits
$\lim_{\mathcal{E} \to \mathcal{E}_{min}+} \eta^{(k)}(\mathcal{E})$ exist, $\lim_{\mathcal{E} \to \infty} (\mathcal{E} - \mathcal{E}_{min})^\lambda \eta^{(k)}(\mathcal{E}) = 0$
for all above mentioned $k$, then

$$a(t) \underset{t \to \infty}{\sim} -\frac{i\hbar}{t} \lambda \, e^{-\frac{i}{\hbar}\mathcal{E}_{min}t} \left[ \alpha_n(t) + \left( -\frac{i\hbar}{t} \right) \alpha_{n-1}(t) \right.$$

$$+ \left( -\frac{i\hbar}{t} \right)^2 \alpha_{n-2}(t)$$

$$\left. + \left( -\frac{i\hbar}{t} \right)^3 \alpha_{n-3}(t) + \dots \right], \tag{36}$$

where

$$\alpha_{n-k}(t) = \sum_{l=0}^{n-k-1} \frac{\Gamma(l+\lambda)}{l!} \ e^{-i\,\frac{\pi(l+\lambda+2)}{2}} \ \eta_0^{(l+k)} \left(\frac{\hbar}{t}\right)^{l+\lambda}, \quad (37)$$

and $\eta_0^{(j)} = \lim_{\mathcal{E} \to \mathcal{E}_{min}+} \eta^{(j)}(\mathcal{E})$, $\eta^{(0)}(\mathcal{E}) = \eta(\mathcal{E})$ and $j = 0, 1, \ldots, n$.

The asymptotic form of $h_u(t)$ for $t \to \infty$ for the $a(t)$ given by the relation (34) looks as follows

$$
\begin{aligned}
h_u^\infty(t) \overset{\text{def}}{=} h_u(t)|_{t \to \infty} \ &= \ \mathcal{E}_{min} - i\frac{\hbar}{t} \\
&\quad - \frac{\omega_0^{(1)}}{\omega_0} \left(\frac{\hbar}{t}\right)^2 + \ldots \ . \ (38)
\end{aligned}
$$

In the more general case of $a(t)$ (see, e.g. (36) ) after some algebra the asymptotic approximation of $a(t)$ can be written as follows

$$a(t) \underset{t \to \infty}{\sim} e^{-i\frac{t}{\hbar}\mathcal{E}_{min}} \sum_{k=0}^{N} \frac{c_k}{t^{\xi+k}}, \qquad (39)$$

where $\xi > 0$ and $c_k$ are complex numbers.

From the relation (39) one concludes that

$$\frac{\partial a(t)}{\partial t} \underset{t \to \infty}{\sim} e^{-i\frac{t}{\hbar}\mathcal{E}_{min}} \left\{ -\frac{i}{\hbar}\mathcal{E}_{min} - \sum_{k=0}^{N} (\xi+k) \frac{c_k}{t^{\xi+k+1}} \right\}. \quad (40)$$

Now let us take into account the relation (11). From this relation and relations (39), (40) it follows that

$$h_\phi(t) \underset{t \to \infty}{\sim} \mathcal{E}_{min} + \frac{d_1}{t} + \frac{d_2}{t^2} + \frac{d_3}{t^3} + \ldots , \qquad (41)$$

where $d_1, d_2, d_3, \ldots$ are complex numbers with negative or positive real and imaginary parts. This means that in the case of the asymptotic
approximation to $a(t)$ of the form (39) the following property holds,

$$\lim_{t \to \infty} h_\phi(t) = \mathcal{E}_{min} < \mathcal{E}_\phi^0. \qquad (42)$$

It seems to be important that results (41) and (42) coincide with the results (25) — (29) obtained for the density $\omega(\mathcal{E})$ given by the formula (18). This means that general conclusion obtained for the other $\omega(\mathcal{E})$ defining unstable states should be similar to those following from (25) — (29).

## 4. Final remarks

Long time properties of the survival probability $|a(t)|^2$ and instantaneous energy $\mathcal{E}_\phi(t)$ are relatively easy to find analytically for times $t \gg \tau_{as}$ even in the general case as it was shown in previous Section. It is much more difficult to analyze these properties analytically in the transition time region where $t \sim \tau_{as}$. It can be done numerically for given models.

The model considered in Sec. 2 and defined by the density $\omega_{BW}(\mathcal{E})$, (18), allows one to find numerically the decay curves and the instantaneous energy $\varepsilon_\phi(t)$ as a function of time $t$. The results presented in this Section have been obtained assuming for simplicity that the minimal energy $\mathcal{E}_{min}$ appearing in the formula (18) is equal to zero, $\mathcal{E}_{min} = 0$. So, all numerical calculations were performed for the density $\tilde{\omega}_{BW}(\mathcal{E})$ given by the following formula

$$\tilde{\omega}_{BW}(\mathcal{E}) = \frac{N}{2\pi} \, \Theta(\mathcal{E}) \, \frac{\gamma_\phi^0}{(\mathcal{E} - \mathcal{E}_\phi^0)^2 + (\frac{\gamma_\phi^0}{2})^2}, \tag{43}$$

for some chosen $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0}$.

Performing calculations particular attention was paid to the form of the probability $|a(t)|^2$, i. e. of the decay curve, and of the instantaneous energy $\varepsilon_\phi(t)$ for times $t$ belonging to the most interesting transition time-region between exponential and nonexponential parts of $|a(t)|^2$, where the following relation corresponding with (31) and (32) takes place,

$$|a_{exp}(t)|^2 \sim |a_{non}(t)|^2, \qquad (44)$$

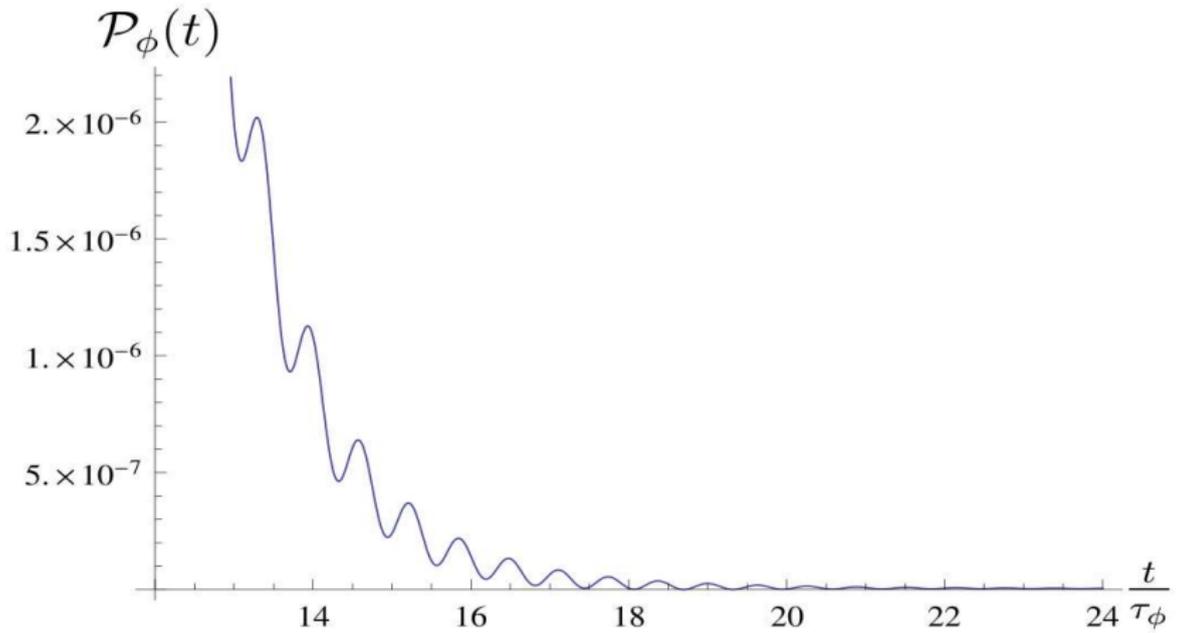where $a_{exp}(t), a_{non}(t)$ are defined by (22). Results are presented graphically below.

Figure: 1. Survival probability $\mathcal{P}_\phi(t) = |a(t)|^2$ in the transition time region. The case $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 10$.
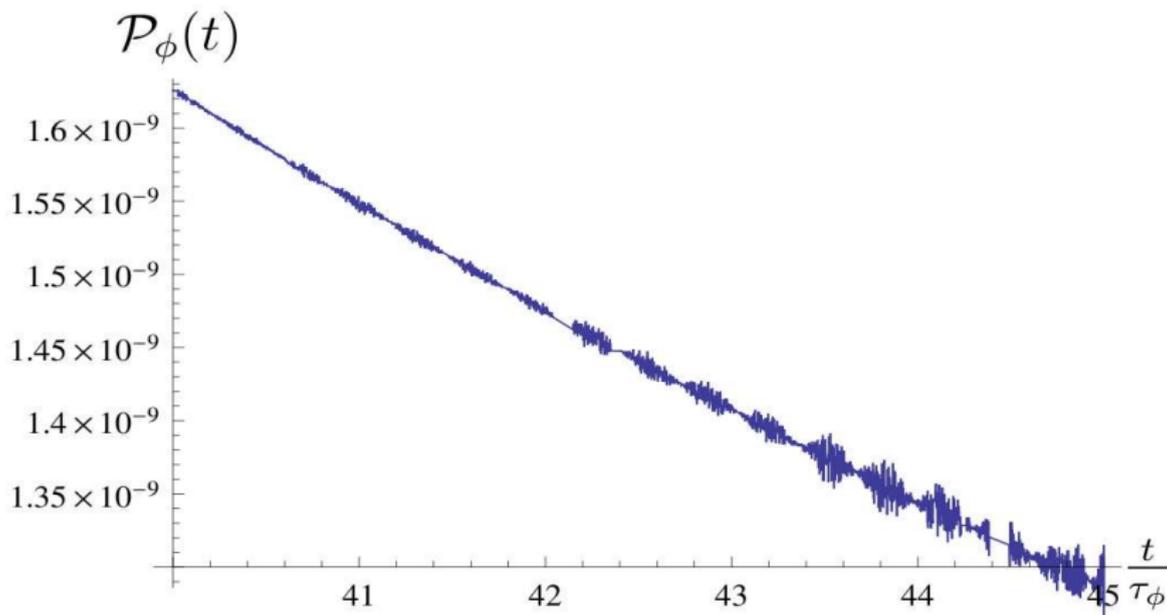
Figure: 2. Survival probability $\mathcal{P}_\phi(t) = |a(t)|^2$ in the transition time region. The case $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 10$.
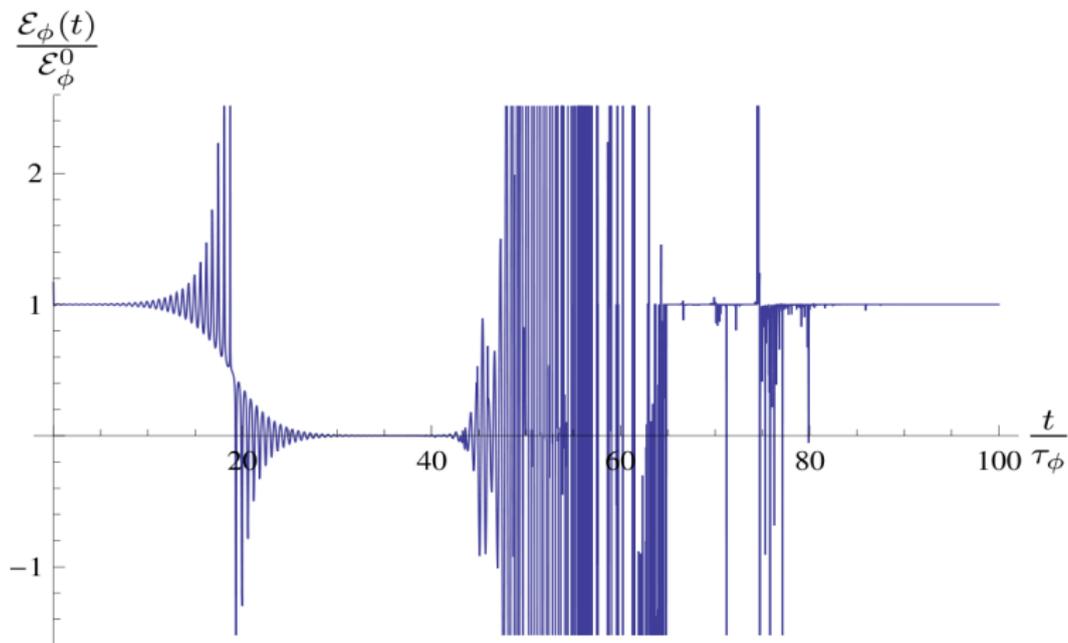
Figure: 3. Instantaneous energy $\mathcal{E}_\phi(t)$ in the transition time region. The case $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 10$.

Figure: 4. Survival probability $\mathcal{P}_\phi(t) = |a(t)|^2$ in the transition time region. The case $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 100$.

Figure: 5. Survival probability $\mathcal{P}_\phi(t) = |a(t)|^2$ in the transition time region. The case $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 100$.

Figure: 6. Instantaneous energy $\mathcal{E}_\phi(t)$ in the transition time region. The case $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 100$.
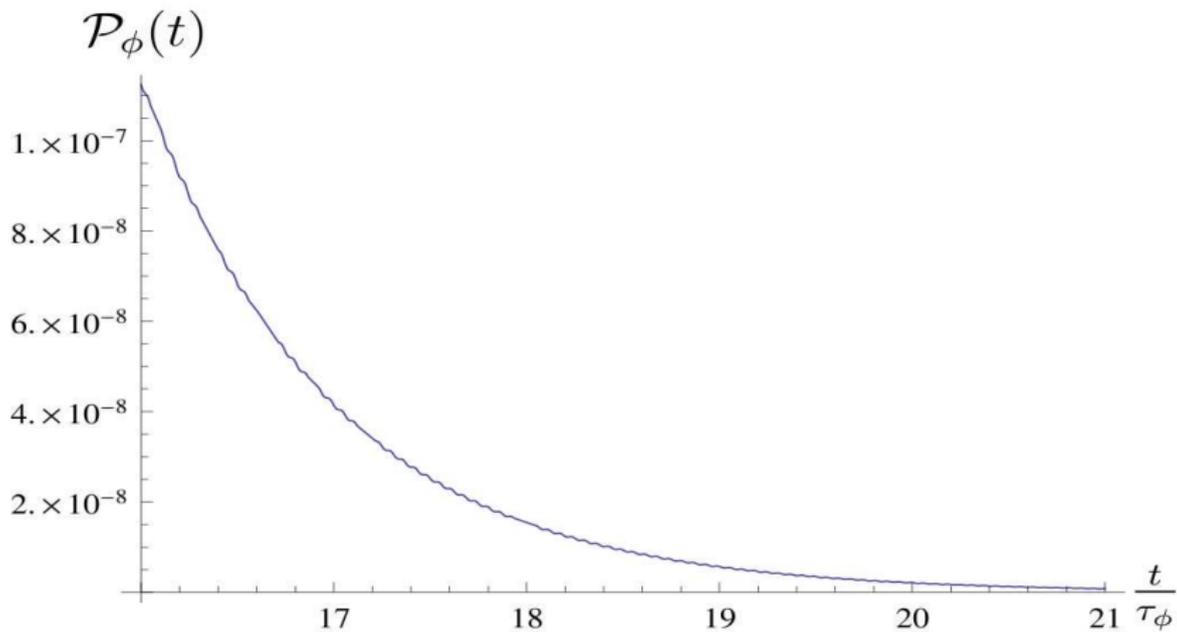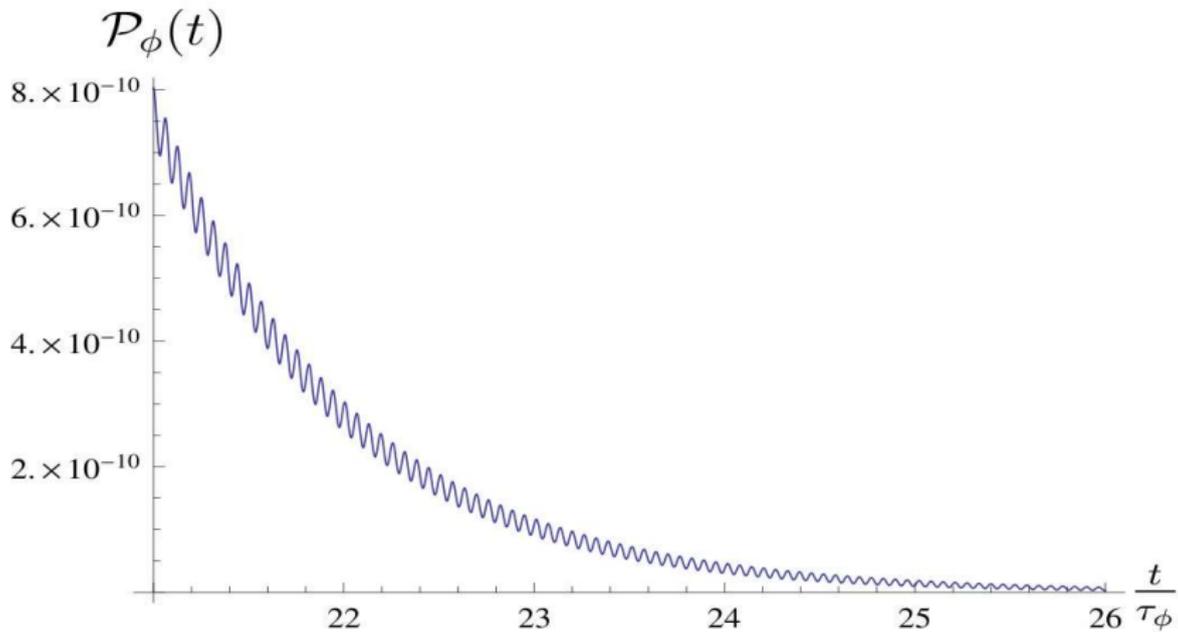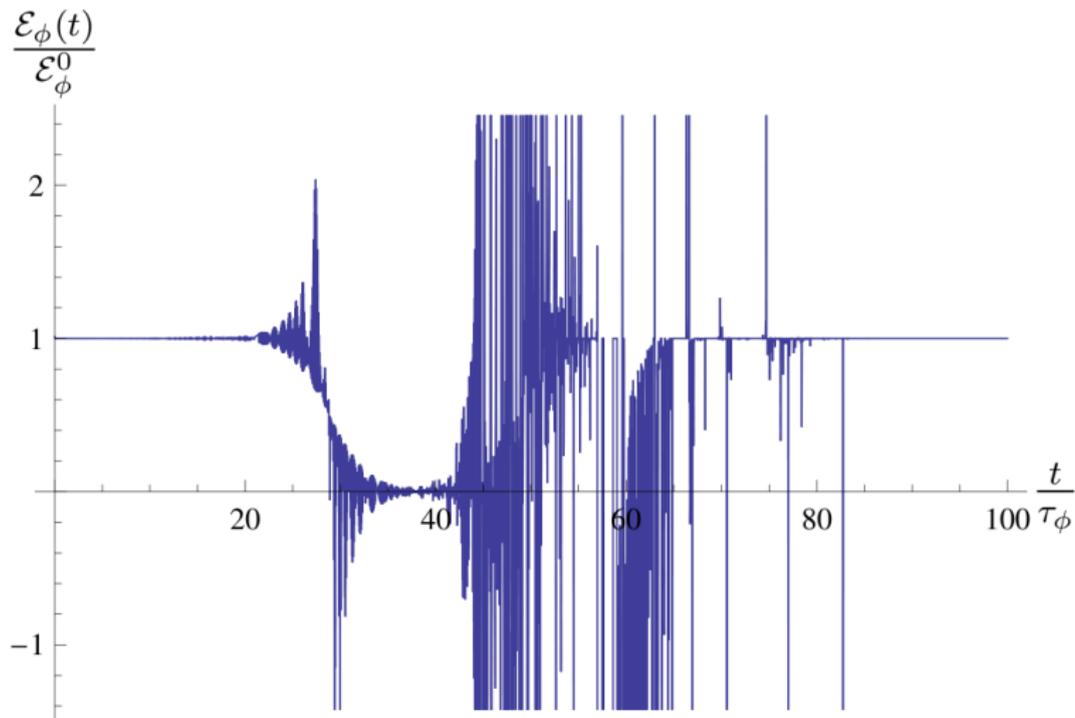
A similar form of a decay curves one meets for a very large class of models defined by energy densities $\omega(\mathcal{E})$ of the following type

$$
\begin{aligned}
\omega(\mathcal{E}) &= \frac{N}{2\pi} \, \Theta(\mathcal{E} - \mathcal{E}_{min}) \, (\mathcal{E} - \mathcal{E}_{min})^{\lambda} \times \\
&\quad \times \frac{\gamma_{\phi}^{0}}{(\mathcal{E} - \mathcal{E}_{\phi}^{0})^{2} + \frac{(\gamma_{\phi}^{0})^{2}}{4}} \, f(\mathcal{E}), \quad\quad (45) \\
&\equiv \omega_{BW}(\mathcal{E}) \, (\mathcal{E} - \mathcal{E}_{min})^{\lambda} \, f(\mathcal{E}),
\end{aligned}
$$

where $\lambda \geq 0$, $f(\mathcal{E})$ is a form–factor — it is a smooth function going to zero as $\mathcal{E} \to \infty$ and it has no threshold and no pole. The asymptotical large time behavior of $a(t)$ is due to the term $(\mathcal{E} - \mathcal{E}_{min})^{\lambda}$ and the choice of $\lambda$ (see Sec. 3). The density $\omega(\mathcal{E})$ defined by the relation (45) fulfills all physical requirements and it leads to the decay curves having a very similar form at transition times region to the decay curves presented above . **The characteristic feature of all these decay curves is the presence of sharp and frequent oscillations at the transition times region** (see Figs (1), (2), (4), (5) ).

**This means that derivatives of the amplitude $a(t)$ may reach extremely large values for some times from the transition time region and the modulus of these derivatives is much larger than the modulus of $a(t)$, which is very small for these times.** This explains why in this time region the real and imaginary parts of $h_\phi(t) \equiv \mathcal{E}_\phi(t) - \frac{i}{2}\,\gamma_\phi(t)$, which can be expressed by the relation (10), ie. by **a large derivative of $a(t)$ divided by a very small $a(t)$,** reach values much larger than the energy $\mathcal{E}_\phi^0$ of the the unstable state measured at times for which the decay curve has the exponential form. **For the model considered we found that, eg. for $\frac{\mathcal{E}_\phi^0}{\gamma_\phi^0} = 10$ and $5\tau_\phi \leq t \leq 60\tau_\phi$ the maximal value of the instantaneous energy equals $\mathcal{E}_\phi(t) = 89,2209\,\mathcal{E}_\phi^0$ and $\mathcal{E}_\phi(t)$ reaches this value for $t \equiv t_{mx,\,10} = 53,94\,\tau_\phi$ and then the survival probability $\mathcal{P}_\phi(t)$ is of order $\mathcal{P}_\phi(t_{mx,\,10}) \sim 10^{-9}$.**

**The question is whether and where this effect can manifest itself.** There are two possibilities to observe the above long time properties of unstable states:

1. One should analyze properties of unstable states having not too long values of $\tau_{as}$.
2. One should find a possibility to observe a suitably large number of events, i.e. unstable particles, created by the same source.

The problem with understanding the properties of **broad resonances** in the scalar sector ($\sigma$ meson problem) discussed in

M. Nowakowski, N. G. Kelkar, *Nishiharima 2004, Penataquark — Proceedings of International Workshop on PENATAQUARK 04, Spring − 8, Hyogo, Japan, 23 −24 July 2004*, pp. 182 − 189.

M. Nowakowski, N. G. Kelkar, *AIP Conf. Proc.* **1030**, (2008), 250 − 255; ArXiv:0807.5103.

where the hypothesis was formulated that this problem could be connected with properties of the decay amplitude in the transition time region, seems to be possible manifestations of this effect and this problem refers to the first possibility mentioned above.

The measured range of possible mass of $\sigma$ meson is very wide, 400 – 1200 MeV. **So one can not exclude the possibility that the masses of some $\sigma$ mesons are measured for times of order their lifetime and some of them for times where their instantaneous energy $\mathcal{E}_\sigma(t)$ is much larger than $\mathcal{E}_\sigma^0$.** This is exactly the case presented in Fig. (3) and Fig. (6). For broad mesons the ratio $\frac{\mathcal{E}_\sigma^0}{\gamma_\sigma^0}$ is relatively small and thus the time $\tau_{as}$ when the above discussed effect occurs appears to be not too long.

**Astrophysical and cosmological processes in which extremely huge numbers of unstable particles are created** seem to be another possibility for the above discussed effect to become manifest. The probability $\mathcal{P}_\phi(t) = |a(t)|^2$ that an unstable particle, say $\phi$, survives up to time $t \sim \tau_{as}$ is extremely small. Let $\mathcal{P}_\phi(t)$ be

$$\mathcal{P}_\phi(t)|_{t \sim \tau_{as}} \sim 10^{-k}, \tag{46}$$

where $k \gg 1$, then there is a chance to observe some of particles $\phi$ survived at $t \sim \tau_{as}$ only if there is a source creating these particles in $\mathcal{N}_\phi$ number such that

$$\mathcal{P}_\phi(t)|_{t \sim \tau_{as}} \, \mathcal{N}_\phi \gg 1. \tag{47}$$

So if a source exists that creates a flux containing

$$\mathcal{N}_\phi \sim 10^l, \tag{48}$$

unstable particles and $l \gg k$ then

**the probability theory states that the number $N_{surv}$ unstable particles**

$$N_{surv} = \mathcal{P}_\phi(t)\big|_{t \sim \tau_{as}} \, \mathcal{N}_\phi \sim 10^{l-k} \gg 0, \qquad (49)$$

**has to survive up to time $t \sim \tau_{as}$.**

Sources creating such numbers of unstable particles are known from cosmology and astrophysics:

▶ The Big Bang.

▶ Processes taking place in galactic nuclei (galactic cores).

▶ Processes taking place inside stars.

▶ Etc.

So let us assume that we have an astrophysical source creating a sufficiently large number of unstable particles in unit of time and emitting a flux of these particles and that this flux is constant or slowly varying in time.

**An example: a flux of neutrons.** From (33) it follows that for the neutron $\tau_{as}^n \sim (250\tau_n - 300\tau_n)$, where $\tau_n \simeq 886$ [s]. If the energies of these neutrons are of order $30 \times 10^{17}$ [eV] then during time $t \sim \tau_{as}^n$ they can reach a distance $d_n \sim 25000$ [ly], that is the distance of about a half of the Milky Way radius. Now if in a unit ot time a suitably large number of neutrons $\mathcal{N}_n$ of the energies mentioned is created by this source then in the distance $d_n$ from the source a number of spherically symmetric space areas (halos) surrounding the source, where neutron instantaneous energies $\mathcal{E}_n(t)$ are much larger than their rest energy $\mathcal{E}_n^0 = \frac{m_n^0 c^2}{\sqrt{1-(\frac{v_n}{c})^2}}$, ($m_n^0$ is the neutron rest mass and $v_n$ denotes its velocity) have to appear (see Fig. (7)). Of course this conclusion holds also for other unstable particles $\phi_\alpha$ produced by this source.
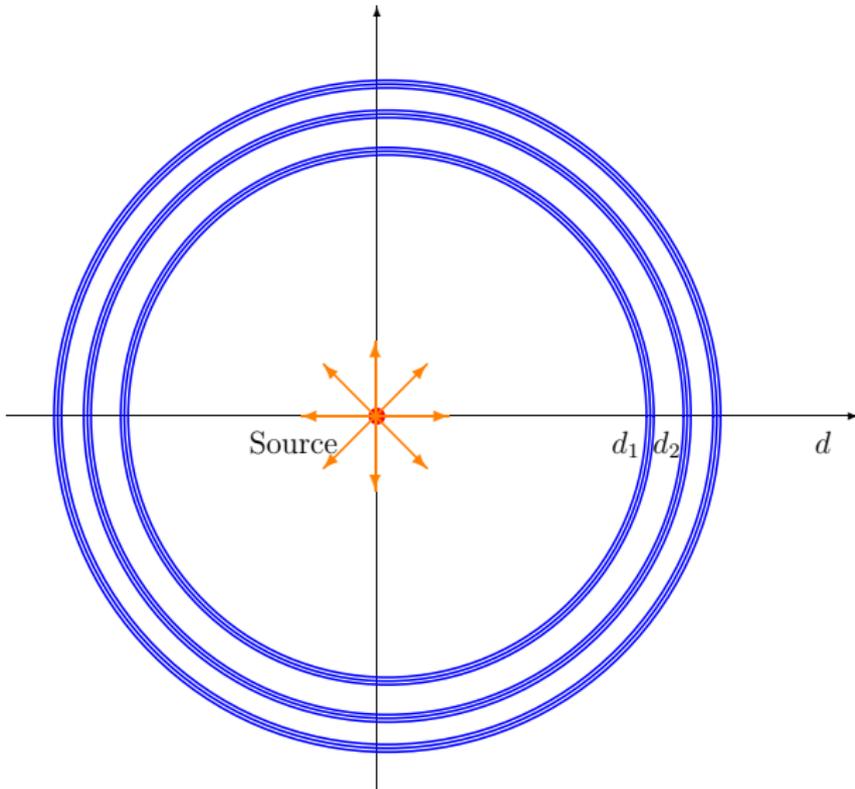
Figure: 7. Halos surrounding a cosmic source of unstable particles.

Every kind of particles $\phi_\alpha$ has its own halos located at distances $d_k^{\phi_\alpha}$,

$$d_k^{\phi_\alpha} \sim v_{\phi_\alpha} \, \tau_{as}^{\phi,\,k}, \quad (k = 1, 2, \ldots),$$

from the source. Radiuses $d_k^{\phi_\alpha}$ of these halos are determined by the particles' velocities $v_{\phi_\alpha}$ and by times $\tau_{as}^{\phi,\,k}$ when instantaneous energies $\mathcal{E}_{\phi_\alpha}(t)$ have local maxima.

**Unstable particles $\phi_\alpha$ forming these halos and having instantaneous energies $\mathcal{E}_{\phi_\alpha}(t) \gg \mathcal{E}_{\phi_\alpha}^0 = m_{\phi_\alpha}^0 \, c^2$ have to interact gravitationally with objects outside of these halos as particles of masses $m_{\phi_\alpha}(t) = \frac{1}{c^2}\mathcal{E}_{\phi_\alpha}(t) \gg m_{\phi_\alpha}^0$.** The possible **observable effects** depend on the astrophysical source of these particles considered.

**1.** If the halos are formed by unstable particles emitted as a result of internal star processes then in the case of very young stars cosmic dust and gases should be attracted by these halos as a result of a gravity attraction. So, **the halos should be a places where the dust and gases condensate.** Thus in the case of very young stars one may consider thehalos as the places where planets are born.

**2.** On the other hand in the case of much older stars **a presence of halos should manifest itself in tiny changes of velocities and accelerations of object moving in the considered planetary star system relating to those calculated without taking into account of the halos presence.**

**3. If the halos are formed by unstable particles emitted by a galaxy core and these particles are such that the ratio $\frac{\mathcal{E}_\phi(t)}{\mathcal{E}_\phi^0}$ is suitably large inside the halos, then rotational velocities of stars rounding the galaxy center outside the halos should differ from those calculated without taking into account the halos.** Thus **the halos may affect the form of rotation curves of galaxies.** (Of course, we do do not assume that the sole factor affecting the form of the rotation curves are these halos).

**4.** Another possible effect is that **the velocities of particles crossing these galactic halos should slightly vary in time due to gravitational interactions,** i. e. they should gain some acceleration. **This should cause charged particles to emit electromagnetic radiation when they cross the halo.**

Note that the above mentioned effects seems to be possible to examine. All these effect are the simple consequence of the fact that the instantaneous energy $\mathcal{E}_{\phi_\alpha}(t)$ of unstable particles becomes large compared with $\mathcal{E}_{\phi_\alpha}^0$ and for some times even extremely large. On the other hand this property of $\mathcal{E}_{\phi_\alpha}(t)$ results form the rigorous analysis of properties of the quantum mechanical survival probability $a(t)$ (see (3) ) and from the assumption that the energy spectrum is bounded from below.

**Thank you**

**ArXiv: 0908.2219**

## References

S. Krylov, V. A. Fock, Zh. Eksp. Teor. Fiz. **17**, (1947), 93.

L. A. Khalfin, Zh. Eksp. Teor. Fiz. **33**, (1957), 1371 [Sov. Phys. — JETP **6**, (1958), 1053].

R. E. A. C. Paley, N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society, New York, 1934.

M. L. Goldberger, K. M. Watson, *Collision Theory*, Willey, New York 1964.

R. G. Newton, *Scattering Theory of Waves and Particles*, Springer, New York 1982.

L. Fonda, G. C. Ghirardii and A. Rimini, Rep. on Prog. in Phys. **41**, (1978), 587.

A. Peres, Ann. Phys. **129**, (1980), 33.

P. T. Greenland, Nature **335**, (1988), 298.

📄 D. G. Arbo, M. A. Castagnino, F. H. Gaioli and S. Iguri, Physica **A 227**, (2000), 469.

📄 J. M. Wessner, D. K. Andreson and R. T. Robiscoe, Phys. Rev. Lett. **29**, (1972), 1126.

📄 E. B. Norman, S. B. Gazes, S. C. Crane and D. A. Bennet, Phys. Rev. Lett. bf 60, (1988), 2246. E. B. Norman, B. Sur, K. T. Lesko, R.-M. Larimer, Phys. Lett. **B 357**, (1995), 521.

📄 J. Seke, W. N. Herfort, Phys. Rev. **A 38**, (1998), 833.

📄 R. E. Parrot, J. Lawrence, Europhys. Lett. **57**, (2002), 632.

📄 J. Lawrence, Journ. Opt. B: Quant. Semiclass. Opt. **4**, (2002), S446.

📄 I. Joichi, Sh. Matsumoto, M. Yoshimura, Phys. Rev. **D 58**, (1998), 045004.

📄 T. Jiitoh, S. Matsumoto, J. Sato, Y. Sato, K. Takeda, Phys Rev. **A 71**, (2005), 012109.

📄 C. Rothe, S. I. Hintschich and A. P. Monkman, Phys. Rev. Lett. **96**, (2006), 163601.

📄 K. Urbanowski, Phys. Rev. **A 50**, (1994), 2847.

📄 K. Urbanowski, Cent. Eur. J. Phys. **7**, (2009), DOI:10.2478/s11534–009–0053–5.

📄 K. M. Sluis, E. A. Gislason, Phys. Rev. **A 43**, (1991), 4581.

📄 *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Ser. No 55, eds. M. Abramowitz nad I. A. Stegun (U.S. GPO, Washington, D.C., 1964).

📄 R. M. Corless, G. H. Gonet, D. E. G. Hare, D. J. Jeffrey and D. E. Khnut, Adv. Comput. Math. **5**, (1996), 329.

📄 F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974. W. Heitler, *The Quantum Theory of Radiation*, Oxford University Press, London 1954 and Dover Publications, New York 1984.

📄 K. Urbanowski, Eur. Phys. J. **D 54**, (2009), 25.

📄 A. Erdelyi, *Asymtotic expansions*, Dover Pub. Inc., New York 1956.

📄 E. T. Copson, *Asymtotic expansions*, Cambridge University Press, 1965.

📄 N. G. Kelkar, M. Nowakowski, K. P. Khemchadani, Phys. Rev. **C 70**, (2004), 024601.

📄 C. Amsler *at al*, Phys. Lett. **B 667**, (2008), 1.

📄 M. Nowakowski, N. G. Kelkar, *Nishiharima 2004, Penataquark — Proceedings of International Workshop on PENATAQUARK 04, Spring – 8, Hyogo, Japan, 23 –24 July 2004*, pp. 182 – 189; arXiv: hep–ph/0411317.

📄 M. Nowakowski, N. G. Kelkar, *AIP Conf. Proc.* **1030**, (2008), 250 – 255; ArXiv:0807.5103.