Introduction to group theory and its representations and some applications to particle physics

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Chapter 1

Introduction into the group theory

1.1 Introduction

In these lectures on group theory and its application to the particle physics there have been considered problems of classification of the particles along representations of the unitary groups, calculations of various characteristics of hadrons, have been studied in some detail quark model. First chapter are dedicated to short introduction into the group theory and theory of group representations. In some datails there are given unitary groups $SU(2)$ and $SU(3)$ which play eminent role in modern particle physics. Indeed $SU(2)$ group is aa group of spin and isospin transforation as well as the base of group of gauge transformations of the electroweak interactons $SU(2) \times U(1)$ of the Salam- Weinberg model. The group $SU(3)$ from the other side is the base of the model of unitary symmetry with three flavours as well as the group of colour that is on it stays the whole edifice of the quantum chromodynamics.

In order to aknowledge a reader on simple examples with the group theory formalism mass formulae for elementary particles would be analyzed in detail. Other important examples would be calculations of magnetic moments and axial-vector weak coupling constants in unitary symmetry and quark models. Formulae for electromagnetic and weak currents are given for both models and problem of neutral currents is given in some detail. Electroweak current of the Glashow-Salam-Weinberg model has been constructed. The notion of colour has been introduced and simple examples with it are given. Introduction of vector bosons as gauge fields are explained. Author would try to write lectures in such a way as to enable an eventual reader to perform calculations of many properties of the elementary particles by oneself.

1.2 Groups and algebras. Basic notions.

Definition of a group

Let be a set of elements G $g_1, g_2, ..., g_n$., with the following properties:

1. There is a multiplication law $g_i g_j = g_l$, and if $g_i, g_j \in G$, then $g_i g_j = g_l \in G$, $i, j, l = 1, 2, ..., n$.

2. There is an associative law $g_i(g_j g_l) = (g_i g_j)g_l$.

3. There exists a unit element e, $eg_i = g_i$, $i = 1, 2, ..., n$.

4. There exists an inverse element g_i^{-1}, $g_i^{-1} g_i = e$, $i = 1, 2, ..., n$.

Then on the set G exists the group of elements $g_1, g_2, ..., g_n$.

As a simple example let us consider rotations on the plane. Let us define a set Φ of rotations on angles ϕ.

Let us check the group properties for the elements of this set.

1. Multiplication law is just a summation of angles: $\phi_1 + \phi_2 = \phi_3 \in \Phi$.

2. Associative law is written as $(\phi_1 + \phi_2) + \phi_3 = \phi_1 + (\phi_2 + \phi_3)$.

3. The unit element is the rotation on the angle $0(+2\pi n)$.

4. The inverse element is the rotation on the angle $-\phi(+2\pi n)$.

Thus rotations on the plane about some axis perpendicular to this plane form the group.

Let us consider rotations of the coordinate axes x, y, z, which define Decartes coordinate system in the 3-dimensional space at the angle θ_3 on the plane xy around the axis z:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\theta_3 & sin\theta_3 & 0 \\ -sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_3(\theta_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (1.1)$$

Let ϵ be an infinitesimal rotation. Let is expand the rotation matrix $R_3(\epsilon)$

in the Taylor series and take only terms linear in ϵ:

$$R_3(\epsilon) = R_3(0) + \frac{dR_3}{d\epsilon}_{\epsilon=0} \epsilon + O(\epsilon^2) = 1 + iA_3\epsilon + O(\epsilon^2), \tag{1.2}$$

where $R_3(0)$ is a unit matrix and it is introduced the matrix

$$A_3 = -i\frac{dR_3}{d\epsilon}_{\epsilon=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3}$$

which we name 'generator of the rotation around the 3rd axis' (or z-axis). Let us choose $\epsilon = \eta_3/n$ then the rotation on the angle η_3 could be obtained by n-times application of the operator $R_3(\epsilon)$, and in the limit we have

$$R_3(\eta_3) = \lim_{n\to\infty} [R_3(\eta_3/n)]^n = \lim_{n\to\infty} [1 + iA_3\eta_3/n]^n = e^{iA_3\eta_3}. \tag{1.4}$$

Let us consider rotations around the axis y:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\theta_2 & 0 & sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & cos\theta_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_2(\theta_2) \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.5}$$

where a generator of the rotation around the axis y is introduced:

$$A_2 = -i\frac{dR_2}{d\epsilon}_{\epsilon=0} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1.6}$$

Repeat it for the axis x:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta_1 & sin\theta_1 \\ 0 & -sin\theta_1 & cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1(\theta_1) \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.7}$$

where a generator of the rotation around the axis xy is introduced:

$$A_1 = -i\frac{dR_1}{d\epsilon}_{\epsilon=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{1.8}$$

Now we can write in the 3-dimensional space rotation of the Descartes coordinate system on the arbitrary angles, for example:

$$R_1(\eta_1)R_2(\eta_2)R_3(\eta_3) = e^{iA_1\eta_1}e^{iA_2\eta_2}e^{iA_3\eta_3}$$

However, usually one defines rotation in the 3-space in some other way, namely by using Euler angles:

$$R(\alpha,\beta,\gamma) = e^{iA_3\alpha}e^{iA_2\beta}e^{iA_3\gamma}$$

functions).

(Usually Cabibbo-Kobayashi-Maskawa matrix is chosen as V_{CKM}=

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}.$$

Here $c_{ij} = cos\theta_{ij}, s_{ij} = sin\theta_{ij}, (i,j = 1,2,3)$, while θ_{ij} - generalized Cabibbo angles. How to construct it using Eqs.(1.1, 1.5, 1.7) and setting $\delta_{13} = 0$?)

Generators $A_l, \quad l = 1,2,3,$ satisfy commutation relations

$$A_i \cdot A_j - A_j \cdot A_i = [A_i, A_j] = i\epsilon_{ijk}A_k, \quad i,j,k = 1,2,3,$$

where ϵ_{ijk} is absolutely antisymmetric tensor of the 3rd rang. Note that matrices $A_l, l = 1,2,3,$ are antisymmetric, while matrices R_l are orthogonal, that is, $R_i^T R_j = \delta_{ij}$, where index T means 'transposition'. Rotations could be completely defined by generators $A_l, l = 1,2,3,$. In other words, the group of 3-dimensional rotations (as well as any continuous Lie group up to discrete transformations) could be characterized by its **algebra** , that is by definition of generators $A_l, l = 1,2,3,$, its linear combinations and commutation relations.

Definition of algebra

L is the Lie algebra on the field of the real numbers if:

(i) L is a linear space over K (for $x \in$ L the law of multiplication to numbers from the set K is defined), (ii) For $x, y \in$L the commutator is defined as $[x,y]$, and $[x,y]$ has the following properties: $[\alpha x, y] = \alpha[x,y], \quad [x, \alpha y] = \alpha[x,y]$ at $\alpha \in K$ and $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$,

$[x, y_1 + y_2] = [x, y_1] + [x, y_2]$ **for all** $x, y \in$**L;**

$[x, x] = 0$ **for all** $x, y \in$**L;**

$[[x,y]z] + [[y,z]x] + [[z,x]y] = 0$ **(Jacobi identity).**

1.3 Representations of the Lie groups and algebras

Before a discussion of representations we should introduce two notions: isomorphism and homomorphism.

Definition of isomorphism and homomorphism

Let be given two groups, G and G'.

Mapping f of the group G into the group G' is called isomorphism or homomorphis

$$If \quad f(g_1g_2) = f(g_1)f(g_2) \quad for \quad any \quad g_1g_2 \in G.$$

This means that if f maps g_1 into g_1' and g_2 в g_2', then f also maps g_1g_2 into $g_1'g_2'$.

However if $f(e)$ maps e into a unit element in G', the inverse in general is not true, namely, e' from G' is mapped by the inverse transformation f^{-1} into $f^{-1}(e')$, named the core (orr nucleus) of the homeomorphism.

If the core of the homeomorphism is e from G such one-to-one homeomorphism is named isomorphism.

Definition of the representation

Let be given the group G and some linear space L. Representation of the group G in L we call mapping T, which to every element g in the group G put in correspondence linear operator $T(g)$ in the space L in such a way that the following conditions are fulfilled:

(1) $T(g_1g_2) = T(g_1)T(g_2)$ **for all** $g_1, g_2 \in G$,

(2) $T(e) = 1$, **where 1 is a unit operator in** L.

The set of the operators $T(g)$ is homeomorphic to the group G.

Linear space L is called the representation space, and operators $T(g)$ are called representation operators, and they map one-to-one L on L. Because of that the property (1) means that the representation of the group G into L is the homeomorphism of the group G into the G^* (group of all linear operators in в L, with one-to-one correspondence mapping of L in L). If the space L is finite-dimensional its dimension is called dimension of the representation T and named as n_T. In this case choosing in the space L a basis $e_1, e_2, ..., e_n$, it is possible to define operators $T(g)$ by matrices of the order n:

$$t(g) = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix},$$

$$T(g)e_k = \sum t_{ij}(g)e_j, \quad t(e) = 1, \quad t(g_1g_2) = t(g_1)t(g_2).$$

The matrix $t(g)$ is called a representation matrix T. If the group G itself consists from the matrices of the fixed order, then one of the simple representations is obtained at $T(g) = g$ (identical or, better, adjoint representation).

Such adjoint representation has been already considered by us above and is the set of the orthogonal 3×3 matrices of the group of rotations O(3) in the 3-dimensional space. Instead the set of antisymmetrical matrices $A_i, i = 1, 2, 3$ forms adjoint representation of the corresponding Lie agebra. It is obvious that upon constructing all the represetations of the given Lie algebra we indeed construct all the represetations of the corresponding Lie group (up to discrete transformations).

By the transformations of similarity подобия $T'(g) = A^-1T(g)A$ it is possible to obtain from $T(g)$ representation $T'(g) = g$ which is equivalent to it but, say, more suitable (for example, representation matrix can be obtained in almost diagonal form).

Let us define a sum of representations $T(g) = T_1(g) + T_2(g)$ and say that a representation is irreducible if it cannot be written as such a sum (For the Lie group representations is definition is sufficiently correct).

For search and classification of the irreducible representation (IR) Schurr's lemma plays an important role.

Schurr's lemma: Let be given two IR's, $t^\alpha(g)$ and $t^\beta(g)$, of the group G. Any matrix B, such that $Bt^\alpha(g) = t^\beta(g)B$ for all $g \in G$ either is equal to 0 (if $t^\alpha(g)$) and $t^\beta(g)$ are not equivalent) or кратна is proportional to the unit matrix λI.

Therefore if $B \neq \lambda I$ exists which commutes with all matrices of the given representation $T(g)$ it means that this $T(g)$ is reducible. Really,, if $T(g)$ is reducible and has the form

$$T(g) = T_1(g) + T_2(g) = \begin{pmatrix} T_1(g) & 0 \\ 0 & T_2(g) \end{pmatrix},$$

then

$$B = \begin{pmatrix} \lambda_1 I^1 & 0 \\ 0 & \lambda_2 I^2 \end{pmatrix} \neq \lambda I$$

and $[T(g), B] = 0$.

For the group of rotations O(3) it is seen that if $[A_i, B] = 0, i = 1, 2, 3$ then $[R^i, B] = 0$,,i.e., for us it is sufficient to find a matrix B commuting with all the generators of the given representation, while eigenvalues of such matrix operator B can be used for classifications of the irreducible representations (IR's). This is valid for any Lie group and its algebra.

So, we would like to find all the irreducible representations of finite dimension of the group of the 3-dimensional rotations, which can be be reduced to searching of all the sets of hermitian matrices $J_{1,2,3}$ satisfying commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

There is only one bilinear invariant constructed from generators of the algebra (of the group): $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, for which $[\vec{J}^2, J_i] = 0, i = 1, 2, 3$. So IR's can be characterized by the index j related to the eigenvalue of the operator $\vec{J}^2$.

In order to go further let us return for a moment to the definition of the representation. Operators $T(g)$ act in the linear n-dimensional space L_n and could be realized by $n \times n$ matrices where n is the dimension of the irreducible representation. In this linear space n-dimensional vectors $\vec{v}$ are defined and any vector can be written as a linear combination of n arbitrarily chosen linear independent vectors $\vec{e}_i$, $\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i$. In other words, the space L_n is spanned on the n linear independent vectors $\vec{e}_i$ forming basis in L_n. For example, for the rotation group $O(3)$ any 3-vector can be definned, as we have already seen by the basic vectors

$$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\vec{x} = xe_x + ye_y + ze_z$. And the 3-dimensional representation (adjoint in this case) is realized by the matrices $R_i, i = 1, 2, 3$. We shall write it now in a different way.

Our problem is to find matrices J_i of a dimension n in the basis of n linear independent vectrs, and we know, first, commutation reltions $[J_i, J_j] = i\epsilon_{ijk}J_k$, and, the second, that IR's can be characterized by $\vec{J}^2$. Besides, it is possible to perform similarity transformation of the Eqs.(1.8,1.6,1.3,) in such a way that one of the matrices, say J_3, becomes diagonal. Then its diagonal elements would be eigenvalues of new basical vectors.

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i \\ 0 & 1+i & 0 \\ -i & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1-i & 0 \\ i & 0 & 1 \end{pmatrix} \tag{1.9}$$

$$UA_2U^{-1} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$U(A_1 + A_3)U^{-1} = 2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$U(A_1 - A_3)U^{-1} = 2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

In the case of the 3-dimensional representation usually one chooses

$$J_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{1.10}$$

$$J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix} \tag{1.11}$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.12}$$

Let us choose basic vectors as

$$|1+1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1 \quad 0> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1-1> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and

$$J_3|1+1>=+|1+1>, \quad J_3|1 \quad 0>=0|1+1>,$$
$$J_3|1-1>=-|1+1>.$$

In the theory of angular momentum these quantities form basis of the representation with the full angular momentum equal to 1. But they could be identified with 3-vector in any space, even hypothetical one. For example, going a little ahead, note that triplet of π-mesons in isotopic space could be placed into these basic vectors:

$$\pi^+, \pi^-, \pi^0 \to |\pi^\pm>=|1 \pm 1>, \quad |\pi^0>=|1 \quad 0>.$$

Let us also write in some details matrices for $J=2$, i.e., for the representation of the dimension $n=2J+1=5$:

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.13}$$

$$J_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{3/2} & 0 & 0 \\ 0 & i\sqrt{3/2} & 0 & -i\sqrt{3/2} & 0 \\ 0 & 0 & i\sqrt{3/2} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix} \tag{1.14}$$

$$J_3 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \tag{1.15}$$

As it should be these matrices satisfy commutation relations $[J_i, J_j] = i\epsilon_{ijk}J_k, i, j, k = 1, 2, 3$, i.e., they realize representation of the dimension 5 of the Lie algebra corresponding to the rotation group $O(3)$.

Basic vectors can be chosen as:

$$|1+2>=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \quad |1+1>=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \quad |1 \quad 0>=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},$$

$$|1-1>=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \quad |1-2>=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}.$$

$J_3|2,+2>=+2|2,+2>$, $J_3|2,+1>=+1|2,+1>$, $J_3|2,0>=0|2,0>$, $J_3|2,-1>=-1|2,-1>$, $J_3|2,-2>=-2|2,-2>$.

Now let us formally put $J=1/2$ although strictly speaking we could not do it. The obtained matrices up to a factor 1/2 are well known Pauli matrices:

$$J_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$J_2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$J_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

These matrices act in a linear space spanned on two basic 2-dimensional vectors

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}.$$

There appears a real possibility to describe states with spin (or isospin) 1/2. But in a correct way it would be possible only in the framework of another group which contains all the representations of the rotation group $O(3)$ plus плюс representations corresponding to states with half-integer spin (or isospin, for mathematical group it is all the same). This group is $SU(2)$.

1.4 Unitary unimodular group SU(2)

Now after learning a little the group of 3-dimensional rotations in which dimension of the minimal nontrivial representation is 3 let us consider more complex group where there is a representation of the dimension 2. For this purpose let us take a set of 2×2 unitary unimodular U,i.e., $U^{\dagger}U = 1$, $detU = 1$. Such matrix U can be written as

$$U = e^{i\sigma_k a_k},$$

$\sigma_k, k = 1, 2, 3$ being hermitian matrices, $\sigma_k^{\dagger} = \sigma_k$, chosen in the form of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $a_k, k = 1, 2, 3$ are arbitrary real numbers. The matrices U form a group with the usual multiplying law for matrices and realize identical (adjoint) represenation of the dimension 2 with two basic 2-dimensional vectors.

Instead Pauli matrices have the same commutation realtions as the generators of the rotation group $O(3)$. Let us try to relate these matrices with a usual 3-dimensional vector $\vec{x} = (x_1, x_2, x_3)$. For this purpose to any vector $\vec{x}$ let us attribute сопоставим a quantity $X = \vec{\sigma}\vec{x}$, :

$$X_b^a = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \quad a, b = 1, 2. \tag{1.16}$$

Its determinant is $detX_b^a = -\vec{x}^2$, that is it defines square of the vector length. Taking the set of unitary unimodular matrices U, $U^{\dagger}U = 1$, $detU = 1$ in 2-dimensional space, let us define

$$X' = U^{\dagger}XU,$$

and $detX' = det(U^{\dagger}XU) = detX = -\vec{x}^2$. We conclude that transformations U leave invariant the vector length and therefore corresponds to rotations in the 3-dimensional space, and note that $\pm U$ correspond to the same rotation. Corresponding algebra $SU(2)$ is given by hermitian matrices $\sigma_k, k = 1, 2, 3$, with the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where $U = e^{i\sigma_k a_k}$.

And in the same way as in the group of 3-rotations $O(3)$ the representation of lowest dimension 3 is given by three independent basis vectors,for example , x,y,z; in $SU(2)$ 2- dimensional representation is given by two independent basic spinors q^α, $\alpha = 1, 2$ which could be chosen as

$$q^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad q^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The direct product of two spinors q^α and q^β can be expanded into the sum of two irreducible representations (IR's) just by symmetrizing and antisymmetrizing the product:

$$q^\alpha \times q^\beta = \frac{1}{2}\{q^\alpha q^\beta + q^\beta q^\alpha\} + \frac{1}{2}[q^\alpha q^\beta - q^\beta q^\alpha] \equiv T^{\{\alpha\beta\}} + T^{[\alpha\beta]}.$$

Symmetric tensor of the 2nd rank has dimension $d^n_{SS} = n(n+1)/2$ and for $n = 2$ $d^2_{SS} = 3$ which is seen from its matrix representation:

$$T^{\{\alpha\beta\}} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}$$

and we have taken into account that $T_{\{21\}} = T_{\{12\}}$.

Antisymmetric tensor of the 2nd rank has dimension $d^n_{AA} = n(n-1)/2$ and for $n = 2$ $d^2_{AA} = 1$ which is also seen from its matrix representation:

$$T^{[\alpha\beta]} = \begin{pmatrix} 0 & T_{12} \\ -T_{12} & 0 \end{pmatrix}$$

and we have taken into account that $T_{[21]} = -T_{[12]}$ and $T_{[11]} = -T_{[22]} = 0$. Or instead in values of IR dimensions:

$$2 \times 2 = 3 + 1.$$

According to this result absolutely antisymmetric tensor of the 2nd rank $\epsilon_{\alpha\beta}$ ($\epsilon_{12} = -\epsilon_{21} = 1$) transforms also as a singlet of the group $SU(2)$ and we can use it to contract $SU(2)$ indices if needed. This tensor also serves to uprise and lower indices of spinors and tensors in the $SU(2)$.

$$\epsilon_{\alpha'\beta'} = u^{\alpha}_{\alpha'} u^{\beta}_{\beta'} \epsilon_{\alpha\beta} :$$

$$\epsilon_{12} = u^1_1 u^2_2 \epsilon_{12} + u^2_1 u^1_2 \epsilon_{21} = (u^1_1 u^2_2 - u^2_1 u^1_2)\epsilon_{12} = DetU \epsilon_{12} = \epsilon_{12}$$

as $DetU = 1$. (The same for ϵ_{21}.)

1.5 $SU(2)$ as a spinor group

Associating q^α with the spin functions of the entities of spin 1/2, $q^1 \equiv |\uparrow\rangle$ and $q^2 \equiv |\downarrow\rangle$ being basis spinors with +1/2 and -1/2 spin projections, correspondingly, (baryons of spin 1/2 and quarks as we shall see later) we can form symmetric tensor $T^{\{\alpha\beta\}}$ with three components

$$T^{\{11\}} = q^1 q^1 \equiv |\uparrow\uparrow\rangle,$$

$$T^{\{12\}} = \frac{1}{\sqrt{2}}(q^1 q^2 + q^2 q^1) \equiv \frac{1}{\sqrt{2}}|(\uparrow\downarrow + \downarrow\uparrow)\rangle,$$

$$T^{\{22\}} = q^2 q^2 \equiv |\downarrow\downarrow\rangle,$$

and we have introduced $1/\sqrt{2}$ to normalize this component to unity.

Similarly for antisymmetric tensor associating again q^α with the spin functions of the entities of spin 1/2 let us write the only component of a singlet as

$$T^{[12]} = \frac{1}{\sqrt{2}}(q^1 q^2 - q^2 q^1) \equiv \frac{1}{\sqrt{2}}|(\uparrow\downarrow - \downarrow\uparrow)\rangle, \tag{1.17}$$

and we have introduced $1/\sqrt{2}$ to normalize this component to unity.

Let us for example form the product of the spinor q^α and its conjugate spinor q_β whose basic vectors could be taken as two rows $(1 \quad 0)$ and $(0 \quad 1)$. Now expansion into the sum of the IR's could be made by subtraction of a trace (remind that Pauli matrices are traceless)

$$q^\alpha \times q_\beta = (q^\alpha q_\beta - \frac{1}{2}\delta^\alpha_\beta q^\gamma q_\gamma) + \frac{1}{2}\delta^\alpha_\beta q^\gamma q_\gamma \equiv T^\alpha_\beta + \delta^\alpha_\beta I,$$

where T^α_β is a traceless tensor of the dimension $d_V = (n^2-1)$ corresponding to the vector representation of the group $SU(2)$ having at $n=2$ the dimension 3; I being a unit matrix corresponding to the unit (or scalar) IR. Or instead in values of IR dimensions:

$$2 \times 2 = 3 + 1.$$

The group $SU(2)$ is so little that its representations $T^{\{\alpha\beta\}}$ and T^α_β corresponds to the same IR of dimension 3 while $T^{[\alpha\beta]}$ corresponds to scalar IR together with $\delta^\alpha_\beta I$. For $n \neq 2$ this is not the case as we shall see later.

One more example of expansion of the product of two IR's is given by the product

$$\begin{aligned} T^{[ij]} \times q^k = \\ = \frac{1}{4}(q^iq^jq^k - q^jq^iq^k - q^iq^kq^j + q^jq^kq^i - q^kq^jq^i + q^kq^iq^j) + \\ \frac{1}{4}(q^iq^jq^k - q^jq^iq^k + q^iq^kq^j - q^jq^kq^i + q^kq^jq^i - q^kq^iq^j) = \\ = T^{[ikj]} + T^{[ik]j} \end{aligned} \tag{1.18}$$

or in terms of dimensions:

$$n(n-1)/2 \times n = \frac{n(n^2-3n+2)}{6} + \frac{n(n^2-1)}{3}.$$

For $n = 2$ antisymmetric tensor of the 3rd rank is identically zero. So we are left with the mixed-symmetry tensor $T^{[ik]j}$ of the dimension 2 for $n = 2$, that is, which describes spin 1/2 state. It can be contracted with the the absolutely anisymmetric tensor of the 2nd rank e_{ik} to give

$$e_{ik}T^{[ik]j} \equiv t^j_A,$$

and t^j_A is just the IR corresponding to one of two possible constructions of spin 1/2 state of three 1/2 states (the two of them being antisymmetrized). The state with the $s_z = +1/2$ is just

$$t^1_A = \frac{1}{\sqrt{2}}(q^1q^2 - q^2q^1)q^1 \equiv \frac{1}{\sqrt{2}}|\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle. \tag{1.19}$$

(Here $q^1 = \uparrow$, $q^2 = \downarrow$.)

The last example would be to form a spinor IR from the product of the symmetric tensor $T^{\{ik\}}$ and a spinor q^j.

$$T^{\{ij\}} \times q^k =$$

$$= \frac{1}{4}(q^i q^j q^k + q^j q^i q^k + q^i q^k q^j + q^j q^k q^i + q^k q^j q^i + q^k q^i q^j) + \tag{1.20}$$

$$\frac{1}{4}(q^i q^j q^k + q^j q^i q^k - q^i q^k q^j - q^j q^k q^i - q^k q^j q^i - q^k q^i q^j) =$$

$$= T^{\{ikj\}} + T^{\{ik\}j}$$

or in terms of dimensions:

$$n(n+1)/2 \times n = \frac{n(n^2+3n+2)}{6} + \frac{n(n^2-1)}{3}.$$

Symmetric tensor of the 4th rank with the dimension 4 describes the state of spin S=3/2, (2S+1)=4. Instead tensor of mixed symmetry describes state of spin 1/2 made of three spins 1/2:

$$e_{ij} T^{\{ik\}j} \equiv T^k_S.$$

T^j_S is just the IR corresponding to the 2nd possible construction of spin 1/2 state of three 1/2 states (with two of them being symmetrized). The state with the $s_z = +1/2$ is just

$$T^1_S = \frac{1}{\sqrt{6}}(e_{12} 2q^1 q^1 q^2 + e_{21}(q^2 q^1 + q^1 q^2) q^1 \equiv \tag{1.21}$$

$$\equiv \frac{1}{\sqrt{6}} |2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle.$$

1.6 Isospin group $SU(2)_I$

Let us consider one of the important applications of the group theory and of its representations in physics of elementary particles. We would discuss classification of the elementary particles with the help of group theory. As a simple example let us consider proton and neutron. It is known for years that proton and neutron have quasi equal masses and similar properties as to strong (or nuclear) interactions. That's why Heisenberg suggested to consider them one state. But for this purpose one should find the group with the (lowest) nontrivial representation of the dimension 2. Let us try (with Heisenberg) to apply here the formalism of the group $SU(2)$ which has as we have seen 2-dimensional spinor as a basis of representation. Let us introduce now a group of isotopic transformations $SU(2)_I$. Now define nucleon as a state with the isotopic spin $I = 1/2$ with two projections (proton with $I_3 = +1/2$ and neutron with $I_3 = -1/2$) in this imagined 'isotopic space' practically in full analogy with introduction of spin in a usual space. Usually basis of the 2-dimensional representation of the group $SU(2)_I$ is written as a isotopic spinor (isospinor)

$$N = \begin{pmatrix} p \\ n \end{pmatrix},$$

what means that proton and neutron are defined as

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Representation of the dimension 2 is realized by Pauli matrices 2×2 $\tau_k, k = 1, 2, 3$ (instead of symbols $\sigma_i, i = 1, 2, 3$ which we reserve for spin $1/2$ in usual space). Note that isotopic operator $\tau^+ = 1/2(\tau_1 + i\tau_2)$ transforms neutron into proton , while $\tau^- = 1/2(\tau_1 - i\tau_2)$ instead transforms proton into neutron.

It is known also isodoublet of cascade hyperons of spin $1/2$ $\Xi^{0,-}$ and masses ~ 1320 MeV. It is also known isodoublet of strange mesons of spin 0 $K^{+,0}$ and masses ~ 490 MeV and antidublet of its antiparticles $\bar{K}^{0,-}$.

And in what way to describe particles with the isospin $I = 1$? Say, triplet of π-mesons π^+, π^-, π^0 of spin zero and negative parity (pseudoscalar mesons) with masses $m(\pi^\pm) = 139,5675 \pm 0,0004$ MeV, $m(\pi^0) = 134,9739 \pm 0,0006$ MeV and practically similar properties as to strong interactions?

In the group of (isotopic) rotations it would be possible to define isotopic vector $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ as a basis (where real pseudoscalar fields $\pi_{1,2}$ are related to charged pions $\pi^\pm$ by formula $\pi^\pm = \pi^1 \pm i\pi^2$, and $\pi^0 = \pi_3$), generators $A_l, l = 1, 2, 3$, as the algebra representation and matrices $R_l, l = 1, 2, 3$ as the group representation with angles θ_k defined in isotopic space. Upon using results of the previous section we can attribute to isotopic triplet **of the real fields** $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in the group $SU(2)_I$ the basis of the form

$$\pi^a_b = \left(\begin{array}{cc} \pi_3/\sqrt{2} & (\pi_1 - i\pi_2)/\sqrt{2} \\ (\pi_1 + i\pi_2)/\sqrt{2} & -\pi_3/\sqrt{2} \end{array} \right) \equiv \left(\begin{array}{cc} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{array} \right),$$

where charged pions are described by **complex fields** $\pi^\pm = (\pi_1 \mp i\pi_2)/\sqrt{2}$. So, pions can be given in isotopic formalism as 2-dimensional matrices:

$$\pi^+ = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad \pi^- = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \quad \pi^0 = \left(\begin{array}{cc} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{array} \right),$$

which form basis of the representation of the dimension 3 whereas the representation itself is given by the unitary unimodular matrices 2×2 U.

In a similar way it is possible to describe particles of any spin with the isospin $I = 1$. Among meson one should remember isotriplet of the vector (spin 1) mesons $\rho^{\pm,0}$ with masses ~ 760 MeV:

$$\left(\begin{array}{cc} \frac{1}{\sqrt{2}}\rho^0 & \rho^+ \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 \end{array} \right). \tag{1.22}$$

Among particles with half-integer spin note, for example, isotriplet of strange hyperons found in early 60's with the spin 1/2 and masses $\sim$1192 MeV $\Sigma^\pm$, Σ^0 which can be writen in the $SU(2)$ basis as

$$\left(\begin{array}{cc} \frac{1}{\sqrt{2}}\Sigma^0 & \Sigma^+ \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 \end{array} \right), . \tag{1.23}$$

Representation of dimension 3 is given by the same matrices U.

Let us record once more that experimentally isotopic spin I is defined as a number of particles $N = (2I + 1)$ similar in their properties, that is, having the same spin, similar masses (equal at the level of percents) and practically identical along strong interactions. For example, at the mass close to 1115

MeV it was found only one particle of spin 1/2 with strangeness S=-1 - it is hyperon Λ with zero electric charge and mass $1115,63 \pm 0,05$ MeV. Naturally, isospin zero was ascribed to this particle. In the same way isospin zero was ascribed to pseudoscalar meson $\eta(548)$.

It is known also triplet of baryon resonances with the spin 3/2, strangness S=-1 and masses $M(\Sigma^{*+}(1385)) = 1382,8 \pm 0,4$ MэB, $M(\Sigma^{*0}(1385)) = 1383,7 \pm 1,0$ MэB, $M(\Sigma^{*-}(1385)) = 1387,2 \pm 0,5$ MэB, (resonances are elementary particles decaying due to strong interactions and because of that having very short times of life; one upon a time the question whether they are 'elementary' was discussed intensively) $\Sigma^{*\pm,0}(1385) \to \Lambda^0\pi^{\pm,0}(88 \pm 2\%)$ or $\Sigma^{*\pm,0}(1385) \to \Sigma\pi(12 \pm 2\%)$ (one can find instead symbol $Y_1^*(1385)$ for this resonance).

It is known only one state with isotopic spin $I = 3/2$ (that is on experiment it were found four practically identical states with different charges) : a quartet of nucleon resonances of spin $J = 3/2$ $\Delta^{++}(1232)$, $\Delta^{+}(1232)$, $\Delta^{0}(1232)$, $\Delta^{-}(1232)$, decaying into nucleon and pion (measured mass difference M_{Δ^+}-M_{Δ^0}= 2,7±0,3 MeV). (We can use also another symbol $N^*(1232)$.) There are also heavier 'replics' of this isotopic quartet with higher spins.

In the system $\Xi^{0,-}\pi^{\pm,0}$ it was found only two resonances with spin 3/2 (not measured yet) $\Xi^{*0,-}$ with masses ~ 1520 MeV, so they were put into isodublet with the isospin $I = 1/2$.

Isotopic formalism allows not only to classify practically the whole set of strongly interacting particles (hadrons) in economic way in isotopic multiplet but also to relate various decay and scattering amplitudes for particles inside the same isotopic multiplet.

We shall not discuss these relations in detail as they are part of the relations appearing in the framework of higher symmetries which we start to consider below.

At the end let us remind Gell-Mann–Nishijima relation between the particle charge Q, 3rd component of the isospin I_3 and hypercharge $Y = S + B$, S being strangeness, B being baryon number (+1 for baryons, -1 for antibaryons, 0 for mesons):

$$Q = I_3 + \frac{1}{2}Y.$$

As Q is just the integral over 4th component of electromagnetic current, it means that the electromagnetic current is just a superposition of the 3rd component of isovector current and of the hypercharge current which is isoscalar.

1.7 Unitary symmetry group SU(3)

Let us take now more complex Lie group, namely group of 3-dimensional unitary unimodular matrices which has played and is playing in modern particle physics a magnificient role. This group has already 8 parameters. (An arbitrary complex 3×3 matrix depends on 18 real parameters, unitarity condition cuts them to 9 and unimodularity cuts one more parameter.)

Transition to 8-parameter group $SU(3)$ could be done straightforwardly from 3-parameter group $SU(2)$ upon changing 2-dimensional unitary unimodular matrices U to the 3-dimensional ones and to the corresponding algebra by changing Pauli matrices $\tau_k, k = 1, 2, 3$ to Gell-Mann matrices $\lambda_\alpha, \alpha = 1, .., 8$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.24}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{1.25}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.26}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.27}$$

$$[\frac{1}{2}\lambda_i, \frac{1}{2}\lambda_j] = i f_{ijk} \frac{1}{2}\lambda_k$$

$f_{123} = 1, f_{147} = \frac{1}{2}, f_{156} = -\frac{1}{2}, f_{246} = \frac{1}{2}, f_{257} = \frac{1}{2}, f_{346} = \frac{1}{2}, f_{367} = -\frac{1}{2}, f_{458} = \frac{\sqrt{3}}{2}, f_{678} = \frac{\sqrt{3}}{2}$.

(In the same way being patient one can construct algebra representation of the dimension n for any unitary group $SU(n)$ of finite n.) These matrices realize 3-dimensional representation of the algebra of the group $SU(3)$ with the basis spinors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Representation of the dimension 8 is given by the matrices 8×8 in the linear space spanned over basis spinors

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$x_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . x_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, x_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

But similar to the case of $SU(2)$ where any 3-vector can be written as a traceless matrix 2×2, also here any 8-vector in $SU(3)$ $X = (x_1, ..., x_8)$ can be put into the form of the 3×3 matrix:

$$X^\alpha_\beta = \frac{1}{\sqrt{2}} \sum\nolimits_{k=1}^{8} \lambda_k x_k = \tag{1.28}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} x_3 + \frac{1}{\sqrt{3}} x_8 & x_1 - i x_2 & x_4 - i x_5 \\ x_1 + i x_2 & -x_3 + \frac{1}{\sqrt{3}} x_8 & x_6 - i x_7 \\ x_4 + i x_5 & x_6 + i x_7 & -\frac{2}{\sqrt{3}} x_8 \end{pmatrix}.$$

In the left upper angle we see immediately previous expression (1.16) from $SU(2)$.

The direct product of two spinors q^α and q^β can be expanded exactly at the same manner as in the case of $SU(2)$ (but now $\alpha, \beta = 1, 2, 3$) into the sum of two irreducible representations (IR's) just by symmetrizing and antisymmetrizing the product:

$$q^\alpha \times q^\beta = \frac{1}{2}\{q^\alpha q^\beta + q^\beta q^\alpha\} + \frac{1}{2}[q^\alpha q^\beta - q^\beta q^\alpha] \equiv T^{\{\alpha\beta\}} + T^{[\alpha\beta]}. \tag{1.29}$$

Symmetric tensor of the 2nd rank has dimension $d_S^n = n(n+1)/2$ and for $n = 3$ $d_S^3 = 6$ which is seen from its matrix representation:

$$T^{\{\alpha\beta\}} = \begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{12} & T^{22} & T^{23} \\ T^{13} & T^{23} & T^{33} \end{pmatrix}$$

and we have taken into account that $T^{\{ik\}} = T^{\{ki\}} \equiv T^{ik}$ $(i \neq k, i, k = 1, 2, 3)$.

Antisymmetric tensor of the 2nd rank has dimension $d_A^n = n(n-1)/2$ and for $n = 3$ $d_A^3 = 3$ which is also seen from its matrix representation:

$$T^{[\alpha\beta]} = \begin{pmatrix} 0 & t^{12} & t^{13} \\ -t^{12} & 0 & t^{23} \\ -t^{13} & -t^{23} & 0 \end{pmatrix}$$

and we have taken into account that $T^{[ik]} = -T^{[ki]} \equiv t^{ik}$ $(i \neq k, i, k = 1, 2, 3)$ and $T^{[11]} = T^{[22]} = T^{[33]} = 0$.

In terms of dimensions it would be

$$n \times n = n(n+1)/2|_{SS} + n(n-1)/2|_{AA} \tag{1.30}$$

or for $n = 3$ $3 \times 3 = 6 + \bar{3}$.

Let us for example form the product of the spinor q^α and its conjugate spinor q_β whose basic vectors could be taken as three rows $(1 \quad 0 \quad 0)$, $(0 \quad 1 \quad 0)$and $(0 \quad 0 \quad 1)$. Now expansion into the sum of the IR's could be made by subtraction of a trace (remind that Gell-Mann matrices are traceless)

$$q^\alpha \times q_\beta = (q^\alpha q_\beta - \frac{1}{n}\delta_\beta^\alpha q^\gamma q_\gamma) + \frac{1}{n}\delta_\beta^\alpha q^\gamma q_\gamma \equiv T_\beta^\alpha + \delta_\beta^\alpha I,$$

where T_β^α is a traceless tensor of the dimension $d_V = (n^2 - 1)$ corresponding to the vector representation of the group $SU(3)$ having at $n = 3$ the dimension 8; I being a unit matrix corresponding to the unit (or scalar) IR. In terms of dimensions it would be $n \times \bar{n} = (n^2 - 1) + 1_n$ or for $n = 3$ $3 \times \bar{3} = 8 + 1$.

At this point we finish for a moment with a group formalism and make a transition to the problem of classification of particles along the representation of the group $SU(3)$ and to some consequencies of it.

Chapter 2

Unitary symmetry and quarks

2.1 Eightfold way. Mass formulae in $SU(3)$.

2.1.1 Baryon and meson unitary multiplets

Let us return to baryons $1/2^+$ and mesons 0^-. As we remember there are 8 particles in each class: 8 baryons: - isodublets of nucleon (proton and neutron) and cascade hyperons $\Xi^{0,-}$, isotriplet of Σ-hyperons and isosinglet Λ, and 8 mesons: isotriplet π, two isodublets of strange K-mesons and isosinglet η. Let us try to write baryons $B(1/2^+)$ as a 8-vector of real fields $B = (B_1, ..., B_8)= (\vec{\Sigma}, N_4, N_5, B_6, B_7, B_8)$, where $\vec{\Sigma}=(B_1, B_2, B_3)= (\Sigma_1, \Sigma_2, \Sigma_3)$. Then the basis vector of the 8-dimensional representation could be written in the matrix form:

$$B^\alpha_\beta = \frac{1}{\sqrt{2}} \sum\nolimits_{k=1}^{8} \lambda_k B_k =$$

$$\frac{1}{\sqrt{2}} \left(\begin{array}{ccc} \Sigma_3 + \frac{1}{\sqrt{3}} B_8 & \Sigma_1 - i\Sigma_2 & N_4 - iN_5 \\ \Sigma_1 + i\Sigma_2 & -\Sigma_3 + \frac{1}{\sqrt{3}} B_8 & B_6 - iB_7 \\ N_4 + iN_5 & B_6 + iB_7 & -\frac{2}{\sqrt{3}} B_8 \end{array} \right) = \tag{2.1}$$

$$\left(\begin{array}{ccc} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda^0 & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{array} \right).$$

At the left upper angle of the matrix we see the previous espression (1.23) from theory of isotopic group $SU(2)$. In a similar way pseudoscalar mesons yield 3×3 matrix

$$P^{\alpha}_{\beta} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \tag{2.2}$$

Thus one can see that the classiification proves to be very impressive: instead of 16 particular particles we have now only 2 unitary multiplets. But what corollaries would be? The most important one is a deduction of mass formulae, that is, for the first time it has been succeded in relating among themselves of the masses of different elementary particles of the same spin.

2.1.2 Mass formulae for octet of pseudoscalar mesons

As it is known, mass term in the Lagrangian for the pseudoscalar meson field described by the wave function P has the form quadratic in mass (to assure that Lagrange-Euler equation for the free point-like meson would yield Gordon equation where meson mass enters quadratically)

$$L^P_m = m^2_P P^2,$$

and for octet of such mesons with all the masses equal (degenerated):

$$L^P_m = m^2_P P^{\alpha}_{\beta} P^{\beta}_{\alpha}$$

(note that over repeated indices there is a sum), while $m_\pi = 140$ MeV,$m_K = 490$ MeV, $m_\eta = 548$ MeV. Gell-Mann proposed to refute principle that Lagrangian should be scalar of the symmetry group of strong intereractions, here unitary group $SU(3)$, and instead introduce symmetry breaking but in such a way as to conserve isotopic spin and strangeness (or hypercharge $Y = S + B$ where B is the baryon charge equal to zero for mesons). For this purpose the symmetry breaking term should have zero values of isospin and hypercharge. Gell-Mann proposed a simple solution of the problem, that is, the mass term should transform as the 33-component of the octet formed by product of two meson octets. (Note that in either meson or baryon octet

33-component of the matrix has zero values of isospin and hypercharge) First it is necessary to extract octet from the product of two octets entering Lagrangian. It is natural to proceed contracting the product $P^{\alpha}_{\eta}P^{\beta}_{\gamma}$ along upper and sub indices as $P^{\alpha}_{\gamma}P^{\gamma}_{\beta}$ or $P^{\gamma}_{\beta}P^{\alpha}_{\gamma}$ and subtract the trace to obtain regular octets

$$M^{\alpha}_{\beta} = P^{\alpha}_{\gamma}P^{\gamma}_{\beta} - \frac{1}{3}P^{\eta}_{\gamma}P^{\gamma}_{\eta}$$

,

$$N^{\alpha}_{\beta} = P^{\gamma}_{\beta}P^{\alpha}_{\gamma} - \frac{1}{3}P^{\eta}_{\gamma}P^{\gamma}_{\eta}$$

$M^{\alpha}_{\alpha} = 0, \quad N^{\alpha}_{\alpha} = 0$ (over repeated indices there is a sum). Components 33 of the octets M^3_3 и N^3_3 would serve us as terms which break symmetry in the mass part of the Lagrangian L^P_m. One should only take into account that in the meson octet there are both particles and antiparticles. Therefore in order to assure equal masses for particles and antiparticles, both symmetry breaking terms should enter Lagrangian with qual coefficients. As a result mass term of the Lagrangian can be written in the form

$$L^P_m = m^2_P P^{\alpha}_{\beta}P^{\beta}_{\alpha} + m^2_{1P}(M^3_3 + N^3_3) =$$

$$= m^2_{0P} P^{\alpha}_{\beta}P^{\beta}_{\alpha} + m^2_{1P}(P^3_{\beta}P^{\beta}_3 + P^{\alpha}_3 P^3_{\alpha}).$$

Taking together coefficients in front of similar bilinear combinations of the pseudoscalar fields we obtain

$$m^2_{\pi} = m^2_{0P}, \quad m^2_K = m^2_{0P} + m^2_{1P}, \quad m^2_{\eta} = m^2_{0P} + \frac{4}{3}m^2_{1P},$$

wherefrom the relation follows immediately

$$4m^2_K = 3m^2_{\eta} + m^2_{\pi}, \quad 4 \times 0,245 = 3 \times 0,30 + 0,02(\mathit{ГэВ})^2.$$

The agreement proves to be impressive taking into account clearness and simplicity of the formalism used.

2.1.3 Mass formulae for the baryon octet $J^P = \frac{1}{2}^+$

Mass term of a baryon B с $J^P = \frac{1}{2}^+$ in the Lagrangian is usually linear in mass (to assure that Lagrange-Euler equation of the full Lagrangian for free point-like baryon would be Dirac equation where baryon mass enter linearly)

$$L_m^B = m_B \bar{B} B.$$

For the baryon octet B_α^β with the degenerated (all equal) masses the corresponding part of the Lagrangian yields

$$L_m^B = m_B \bar{B}_\beta^\alpha B_\alpha^\beta,$$

But real masse are not degenerated at all: $m_N \sim 940, \quad m_\Sigma \sim 1192, \quad m_\Lambda \sim 1115 \quad m_\Xi \sim 1320$ (in MeV). Also here Gell-Mann proposed to introduce mass breaking through breaking in a definite way a symmetry of the Lagrangian:

$$L_m^B = m_0 \bar{B}_\beta^\alpha B_\alpha^\beta + m_1 \bar{B}_\beta^3 B_3^\beta + m_2 \bar{B}_3^\alpha B_\alpha^3).$$

Note that here there are two terms with the 33-component as generally speaking $m_1 \neq m_2$. (While mesons and antimesons are in the same octet, baryons and antibaryons forms two different octets) Then for particular baryons we have:

$$p = B_3^1, \quad n = B_3^2 \quad m_p = m_n = m_0 + m_1$$

$$\Sigma^+ = B_2^1, \quad \Sigma^- = B_1^2 \quad m_{\Sigma^{\pm,0}} = m_0$$

$$-\frac{2}{\sqrt{6}}\Lambda_0 = B_3^3, \quad m_\Lambda = m_0 + \frac{2}{3}(m_1 + m_2),$$

$$\Xi_- = B_1^3, \quad \Xi^0 = B_2^3 \quad m_{\Xi^{-,0}} = m_0 + m_2$$

The famous Gell-Mann-Okubo mass relation follows immediately:

$$2(m_N + m_\Xi) = m_\Sigma + 3m_\Lambda, \quad 4520 = 4535.$$

(Values at the left-hand side (LHS) and right-hand side (RHS) are given in MeV.) The agreement with experiment is outstanding which has been a stimul to further application of the unitary Lie groups in particle physics.

2.1.4 Nonet of the vector meson and mass formulae

Mass formula for the vector meson is the same as that for pseudoscalar ones (in this model unitary space do not depends on spin indices).

But number of vector mesons instead of 8 is 9, therefore we apply this formula taking for granted that it is valid here, to find mass of the isoscalar vector meson ω_0 of the octet:

$$m_{\omega_0}^2 = \frac{1}{3}(4m_{K^*}^2 - m_\rho^2),$$

wherefrom $m_{\omega_0} = 930$ MeV. But there is no such isoscalar vector meson of this mass. Instead there are a meson ω with the mass $m_\omega = 783$ MeV and a meson ϕ with the mass $m_\phi = 1020$ Mev. Okubo was forced to introduce nonet of vector mesons as a direct sum of the octet and the singlet

$$V_\beta^\alpha = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix} + \\ + \begin{pmatrix} \frac{1}{\sqrt{3}}\phi_0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}\phi_0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}}\phi_0 \end{pmatrix} \tag{2.3}$$

which we write going a little forward as

$$V_\beta^\alpha = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix} \tag{2.4}$$

$$\begin{pmatrix} \omega \\ \phi \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix} \begin{pmatrix} \omega_8 \\ \phi_0 \end{pmatrix},$$

where $\sqrt{\frac{1}{3}}$ is essentially $cos\theta_{\omega\phi}$, $\theta_{\omega\phi}$ being the angle of ideal mixing of the octet and singlet states with $I = 0, \quad S = 0$. Let us stress once more that introduction of this angle was caused by discrepancy of the mass formula for vector mesons with experiment.

"Nowadays (1969!) there is no serious theoretical basis to treat V_9 (vector nonet–V.Z.) as to main quantity never extracting from it ω_0 (our ϕ_0–V.Z.) as SpV_9 so Okubo assertion should be seen as curious but not very profound observation." S.Gasiorowicz, "Elementary particles physics", John Wiley & Sons, Inc. NY-London-Sydney, 1968.

We will see a little further that the Okubo assertion is not only curious but also very profound.

2.1.5 Decuplet of baryon resonances with $J^P = \frac{3}{2}^+$ and its mass formulae

Up to 1963 nine baryon resonances with $J^P = \frac{3}{2}^+$ were established: isoquartet (with $I = \frac{3}{2}$) $\Delta(1232) \to N+\pi$; isotriplet $\Sigma^*(1385) \to \Lambda+\pi, \Sigma+\pi$; isodublet $\Xi^*(1520) \to \Xi + \pi$. But in $SU(3)$ there is representation of the dimension 10, an analogue to IR of the dimension 4 in $SU(2)$ (with $I = \frac{3}{2}$) which is given by symmetric tensor of the 3rd rank (what with symmetric tensor describing the spin state $J^P = \frac{3}{2}^+$ yields symmetric wave function for the particle of half-integer spin!!!). It was absent a state with strangeness -3 which we denote as Ω. With this state decuplet can be written as:

$$
\begin{array}{c}
\Delta^-_{222} \quad \Delta^0_{221} \quad \Delta^+_{211} \quad \Delta^{++}_{111} \\
\Sigma^{*-}_{223} \quad \Sigma^{*0}_{123} \quad \Sigma^{*+}_{113} \\
\Xi^{*-}_{233} \quad \Xi^{*0}_{133} \\
? \quad \Omega^-_{333} \quad ?
\end{array}
$$

Mass term of the Lagrangian for resonance decuplet $B^{*\alpha\beta\gamma}$ following Gell-Mann hypothesis about octet character of symmetry breaking of the Lagrangian can be written in a rather simple way:

$$L^{B^*}_{M^*} = M^*_0 \bar{B}^*_{\alpha\beta\gamma} B^{\alpha\beta\gamma} + M^*_1 \bar{B}^*_{3\beta\gamma} B^{3\beta\gamma}.$$

Really from unitary wave functions of the decuplet of baryon resonances $B^{*\alpha\beta\gamma}$ and corresponding antidecuplet $\bar{B}_{*\alpha\beta\gamma}$ due to symmetry of indices it is possible to constract an octet in a unique way. The result is:

$$M_\Delta = M^*_0$$

$$
\begin{aligned}
M_{\Sigma^*} &= M^*_0 + M^*_1 \\
M_{\Xi^*} &= M^*_0 + 2M^*_1 \\
M_{\Omega^-} &= M^*_0 + 3M^*_1
\end{aligned}
$$

Mass formula of this kind is named equidistant. It is valid with sufficient accuracy, the step in mass scale being around 145 MeV. But in this case the predicted state of strangeness -3 and mass $(1530 + 145) = 1675$ MeV cannot be a resonance as the lighest two-particle state of strangeness -3 would be $(\Xi(1320)K(490))$ with the mass 1810 MeV! It means that if it exists it should be a particle stable relative to strong interactions and should decay through weak interactions in a cascade way loosing strangeness -1 at each step.

This prediction is based entirely on the octet symmetry breaking of the Lagrangian mass term of the baryon decuplet $B^{*\alpha\beta\gamma}$.

Particle with strangeness -3 Ω^- was found in 1964, its mass turned out to be $(1672, 5 \pm 0, 3)$ MeV coinciding exactly with $SU(3)$ prediction!

It was a triumph of unitary symmetry! The most of physicists believed in it from 1964 on. (By the way the spin of the Ω^- hyperon presumably equal to 3/2 has never been measured.)

2.2 Praparticles and hypothesis of quarks

Upon comparing isotopic and unitary symmetry of elementary particles one can note that in the case of isotopic symmetry the lowest possible IR of the dimension 2 is often realized which has the basis $(1 \quad 0)^T$, $(0 \quad 1)^T$; along this representation, for example, N, Ξ, K, Ξ^*, K^* transform, however at the same time unitary multiplets of hadrons begin from the octet (analogue of isotriplet in $SU(2)_I$).

The problem arises whether in nature the lowest spinor representations are realized. In other words whether more elementary particles exist than hadrons discussed above?

For methodical reasons let us return into the times when people was living in caves, used telegraph and vapor locomotives and thought that π-mesons were bounded states of nucleons and antinucleons and try to understand in what way one can describe these states in isotopic space.

Let us make a product of spinors $N^a, \bar{N}_b, a, b = 1, 2$, and then subrtract and add the trace $\bar{N}_c N^c, c = 1, 2, 3$, expanding in this way a product of two irreducible representations (IR) (two spinors) into the sum of IR's:

$$\bar{N}_b \times N^a = (\bar{N}_b N^a - \frac{1}{2}\delta^a_b \bar{N}_c N^c) + \frac{1}{2}\delta^a_b \bar{N}_c N^c, \tag{2.5}$$

what corresponds to the expansion in terms of isospin $\frac{1}{2} \times \frac{1}{2} = 1 + 0$, or (in terms of IR dimensions) $2 \times 2 = 3 + 1$. In matrix form

$$(\bar{p}, \bar{n}) \times \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} (\bar{p}p - \bar{n}n)/2 & \bar{n}p \\ \bar{p}n & -(\bar{p}p - \bar{n}n)/2 \end{pmatrix} + $$
$$+ \begin{pmatrix} (\bar{p}p + \bar{n}n)/2 & 0 \\ 0 & (\bar{p}p + \bar{n}n)/2 \end{pmatrix}, \tag{2.6}$$

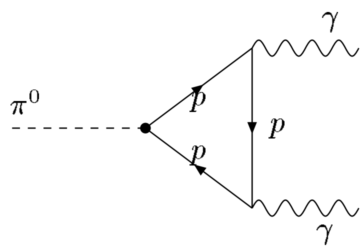
which we identify for the J=0 state of nucleon and antinucleon spins and zero orbital angular momentum with the pion isotriplet and isosinglet η:

$$\pi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}}\eta^0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\eta^0 \end{pmatrix},$$

while for for the J=1 state of nucleon and antinucleon spins and zero orbital angular momentum with the ρ-meson isotriplet and isosinglet ω:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 & \rho^+ \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}}\omega^0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\omega^0 \end{pmatrix}.$$

This hypothesis was successfully used many times. For example, within this hypothesis the decay π^0 into two γ-quanta was calculated via nucleon loop,



The answer coincided exactly with experiment what was an astonishing achievement. Really, mass of two nucleons were so terribly larger than the mass of the π^0-meson that it should be an enormous bounding energy between nucleons. However the answer was obtained within assumption of quasi-free nucleons (1949, one of the achievements of Feynman diagram technique applied to hadron decays.)

With observation of hyperons number of fundamental baryons was increased suddenly. So first composite models arrived. Very close to model of unitary symmetry was Sakata model with proton, neutron and Λ hyperon as a fundamental triplet. But one was not able to put baryon octet into such model. However an idea to put something into triplet remains very attractive.

2.3 Quark model. Mesons in quark model.

Model of quarks was absolutely revolutionary. Gell-Mann and Zweig in 1964 assumed that there exist some praparticles transforming along spinor representation of the dimension 3 of the group $SU(3)$ (and correspondingly antipraparticles transforming along conjugated spinor representation of the dimension 3), and all the hadrons are formed from these fundamental particles. These praparticles named quarks should be fermions (in order to form existing baryons), and let it be fermions with $J^P = \frac{1}{2}^+$ $q^\alpha, \alpha = 1,2,3, q^1 = u, q^2 = d, q^3 = s$. Note that because one needs at least three quarks in order to form baryon of spin 1/2, electric charge as well as hypercharge turn out to be дробными non-integer(!!!) which presented 40 years ago as open heresy and for many of us really unacceptable one.

Quarks should have the following quantum numbers:

	Q	I	I_3	Y=S+B	B
u	2/3	1/2	1/2	1/3	1/3
d	-1/3	1/2	-1/2	1/3	1/3
s	-1/3	0	0	-2/3	1/3

in order to assure the right quantum numbers of all 8 known baryons of spin 1/2 (2.1): $p(uud)$, $n(ddu)$, $\Sigma^+(uus)$, $\Sigma^0(uds)$, $\Sigma^-(dds)$, $\Lambda(uds)$, $\Xi^0(ssu)$, $\Xi^-(ssd)$. In more details we shall discuss baryons a little further in another section in order to maintain the continuity of this talk.

First let us discuss meson states. We can try to form meson states out of quarks in complete analogy with our previous discussion on nucleon-antinucleon states and Eqs.(2.5,2.6):

$$\bar{q}_\beta \times q^\alpha = (\bar{q}_\beta q^\alpha - \frac{1}{3}\delta^\alpha_\beta \bar{q}_\gamma q^\gamma) + \frac{1}{3}\delta^\alpha_\beta \bar{q}_\gamma q^\gamma),$$

$$(\bar{u}, \bar{d}, \bar{s}) \times \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} \bar{u}u & \bar{d}u & \bar{s}u \\ \bar{u}d & \bar{d}d & \bar{s}d \\ \bar{u}s & \bar{d}s & \bar{s}s \end{pmatrix} =$$

$$\begin{pmatrix} D_1 & \bar{d}u & \bar{s}u \\ \bar{u}d & D_2 & \bar{s}d \\ \bar{u}s & \bar{d}s & D_3 \end{pmatrix} + \frac{1}{3}(\bar{u}u + \bar{d}d + \bar{s}s)I, \tag{2.7}$$

where

$$\begin{aligned} D_1 &= \bar{u}u - \frac{1}{3}\bar{q}q = \frac{1}{2}(\bar{u}u - \bar{d}d) + \frac{1}{6}(\bar{u}u + \bar{d}d - 2\bar{s}s) = \\ &= \frac{1}{2}\bar{q}\lambda_3 q + \frac{1}{2\sqrt{3}}\bar{q}\lambda_8 q, \\ D_2 &= \bar{d}d - \frac{1}{3}\bar{q}q = -\frac{1}{2}(\bar{u}u - \bar{d}d) + \frac{1}{6}(\bar{u}u + \bar{d}d - 2\bar{s}s) = \\ &= -\frac{1}{2}\bar{q}\lambda_3 q + \frac{1}{2\sqrt{3}}\bar{q}\lambda_8 q, \\ D_3 &= \bar{s}s - \frac{1}{3}\bar{q}q = -\frac{2}{6}(\bar{u}u + \bar{d}d - 2\bar{s}s) = -\frac{1}{\sqrt{3}}\bar{q}\lambda_8 q. \end{aligned}$$

We see that the traceless matrix obtained here could be identified with the meson octet $J^P = 0^-$ (S-state), the quark structure of mesons being:

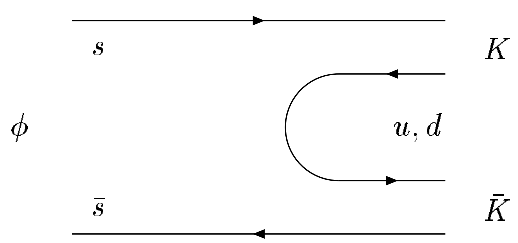
$$\begin{gathered} \pi^- = (\bar{u}d), \quad \pi^+ = (\bar{d}u), \\ K^- = (\bar{u}s), \quad K^+ = (\bar{s}u), \\ \pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d), \\ K^0 = (\bar{s}d), \quad \bar{K}^0 = (\bar{d}s), \\ \eta = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s). \end{gathered}$$

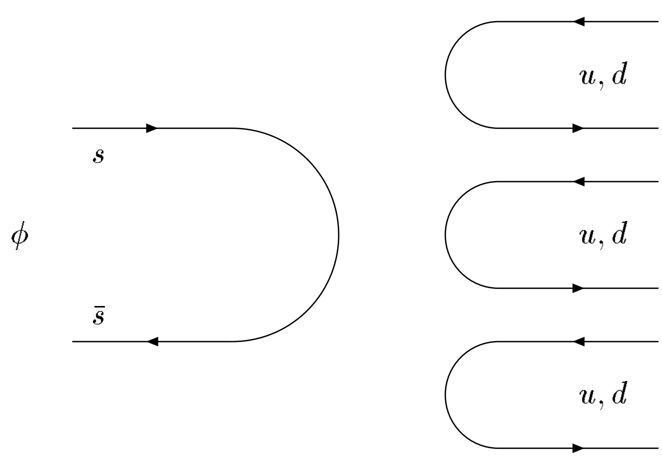
In a similar way nonet of vector mesons Eq.(2.10) can be constructed. But as they are in 9, we take straightforwardly the first expression in Eq.(2.7) with the spins of quarks forming $J = 1$ (always S-state of quarks):

$$\begin{aligned} (\bar{u}, \bar{d}, \bar{s})_\uparrow \times \begin{pmatrix} u \\ d \\ s \end{pmatrix}_\uparrow &= \begin{pmatrix} \bar{u}u & \bar{d}u & \bar{s}u \\ \bar{u}d & \bar{d}d & \bar{s}d \\ \bar{u}s & \bar{d}s & \bar{s}s \end{pmatrix}_{J=1} = \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix} \end{aligned} \tag{2.8}$$

This construction shows immediately a particular structure of the ϕ meson : it contains only strange quarks!!!

Immediately it becomes clear more than strange character of its decay channels. Namely while ω meson decays predominantly into 3 pions, the ϕ meson practically does not decay in this way $(2,5 \pm 09)\%$ although energetically it is very 'profitable' and, instead, decays into the pair of kaons ($(49,1 \pm 0,9)\%$ into the pair K^+K^- and $(34,3 \pm 0,7)\%$ into the pair $K_L^0 K_S^0$). This strange experimental fact becomes understandable if we expose quark diagrams at the simplest level:





Thus we have convinced ourselves that Okubo note on nonet was not only curious but also very profound.

We see also that experimental data on mesons seem to support existence of three quarks.

But is it possible to estimate effective masses of quarks? Let us assume that ϕ-meson is just made of two strange quarks, that is, $m_s = m(\phi)/2 \equiv 510$ MeV. An effective mass of two light quarks let us estimate from nucleon mass as $m_u = m_d = M_p/s \equiv 310$ MeV. These masses are called constituent ones. Now let us look how it works.

$$M_{p(uu,d)} = M_{n(dd,u)} = 930 \quad \text{Mev} \qquad (\text{input})(\sim 940)\text{exp}$$

$$M_{\Sigma(qq,s)} = 1130 \quad \text{Mev} \qquad (\sim 1192)\text{exp}$$

$$M_{\Lambda(uds)} = 1130, \quad \text{Mev} \qquad (\sim 1115)\text{exp}$$

$$M_{\Xi(ss,q)} = 1330 \quad \text{Mev} \qquad (\sim 1320)\text{exp}$$

(There is one more very radical question:

Whether quarks exist at all?

From the very beginning this question has been the object of hot discussions. Initially Gell-Mann seemed to consider quarks as some suitable mathematical object for particle physics. Nowadays it is believed that quarks are as real as any other elementary particles. In more detail we discuss it a little later.)

"Till Parisina's fatal charms
Again attracted every eye..."–

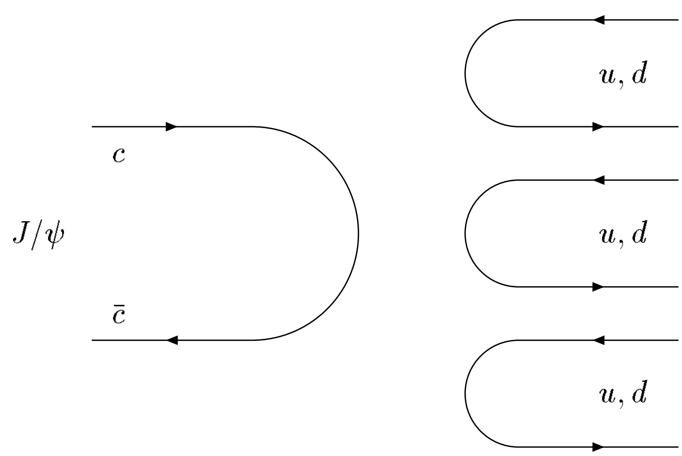
Lord Byron, 'Parisina'

2.4 Charm and its arrival in particle physics.

Thus, there exist three quarks!

During 10 years everybody was thinking in this way. But then something unhappened happened.

In november 1974 on Brookehaven proton accelerator with max proton energy 28 GeV (USA) and at the electron-positron rings SPEAR(SLAC,USA) it was found a new particle– J/ψ vector meson decaying in pions in the hadron channel at surprisingly large mass 3100 MeV and surprisingly long mean life and, correspondingly, small width at the level of 100 KeV although for hadrons characteristic widths oscillate between 150 MeV for rho meson, 8 Mev for ω and 4 MeV for ϕ meson. Taking analogy with suppression of the 3-pion decay of the vector ϕ meson which as assumed is mainly $(\bar{s}s)$, state the conclusion was done that the most simple solution would be hypothesis of existence of the 4th quark with the new quantum number "charm". In this case $J/\psi(3100)$ is the $(\bar{c}c)$ vector state with hidden charm.

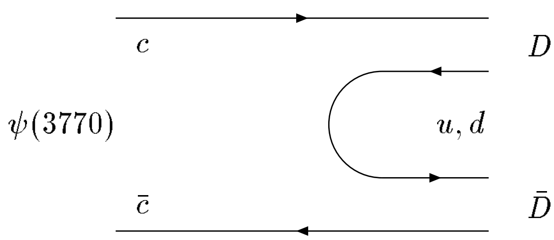


And what is the mass of charm quark? Let us make a bold assumption that as in the case of the $\phi(1020)$ meson where mass of a meson is just double value of the ('constituent') strange quark mass (500 MeV) (so called 'constituent' masses of quarks u and d are around 300 Mev as we have see) mass of the charm quark is just half of that of the $J/\psi(3100)$ particle that is around 1500 MeV (more than 1.5 times proton mass!).

But introduction of a new quark is not so innocue. One assumes with this hypothesis existence of the whole family of new particles as mesons with the charm with quark content ($\bar{u}c$), ($\bar{c}u$), ($\bar{d}c$), ($\bar{c}d$), with masses around (1500+300=1800) MeV at least and also ($\bar{s}c$) и ($\bar{c}s$) with masses around (1500+500=2000) MeV. As it is naturally to assume that charm is conserved in strong interaction as strangeness does, these mesons should decay due to weak interaction loosing their charm. For simplicity we assume that masses of these mesons are just sums of the corresponding quark masses.

Let us once more use analogy with the vector $\phi(1020)$ meson main decay channel of which is the decay $K(490)$ $\bar{K}(490)$ and the production of this meson with the following decay due to this channel dominates processes at the electron-positron rings at the total energy of 1020 MeV.

If analogue of $J/\psi(3100)$ with larger mass exists, namely, in the mass region $2\times$(1500+300)=3600 MeV, in this case such meson should decay mostly to pairs of charm mesons. But such vector meson $\psi(3770)$ was really found at the mass 3770 MeV and the main decay channel of it is the decay to two new particles– pairs of charm mesons $D^0(1865)\bar{D}^0(1865)$ or $D^+(1870)D^-(1870)$ and the corresponding width is more than 20 MeV!!!



New-found charm mesons decay as it was expected due to weak interaction what is seen from a characteristic mean life at the level of $10^{-12} - 10^{-13}$ s.

Unitary symmetry group for particle classification grows to $SU(4)$. It is to note that it would be hardly possible to use it to construct mass formulae as $SU(3)$ because masses are too different in 4-quark model. In any case a problem needs a study.

In the model of 4 quarks (4 flavors as is said today) mesons would transform along representations of the group $SU(4)$ contained in the direct product of the 4-dimensional spinors 4 and $\bar{4}$,: $\bar{4}\times 4 = 15+1$, or

$$\bar{q}_\beta \times q^\alpha = (\bar{q}_\beta q^\alpha - \frac{1}{4}\delta^\alpha_\beta \bar{q}_\gamma q^\gamma) + \frac{1}{4}\delta^\alpha_\beta \bar{q}_\gamma q^\gamma),$$

where now $\alpha, \beta, \gamma = 1, 2, 3, 4$.

$$P^\alpha_\beta = \begin{pmatrix} \eta(2980) & D^0(1865) & D^+(1870) & F^+(1969) \\ D^0(1865) & \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ D^-(1870) & \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ F^-(1969) & K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \tag{2.9}$$

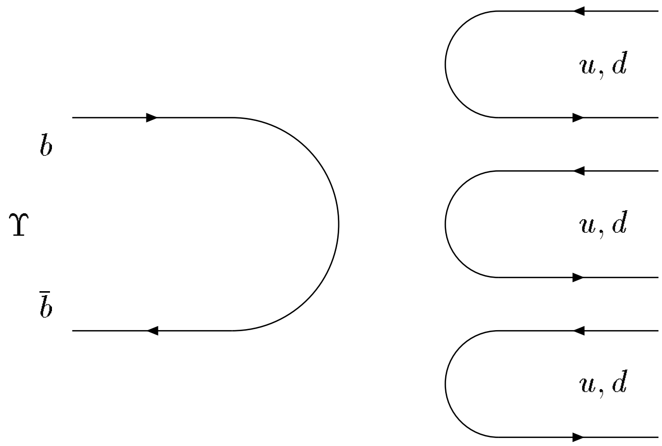
$$V^\alpha_\beta = \begin{pmatrix} \psi(3100) & D^{*0}(2007) & D^{*+}(2010) & F^{*+}(2112) \\ D^{*0}(2007) & \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & \rho^+ & K^+ \\ D^{*-}(2010) & \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & K^{*0} \\ F^{*-}(2112) & K^{-*} & \bar{K}^{*0} & \phi \end{pmatrix} \tag{2.10}$$

2.5 The fifth quark. Beauty.

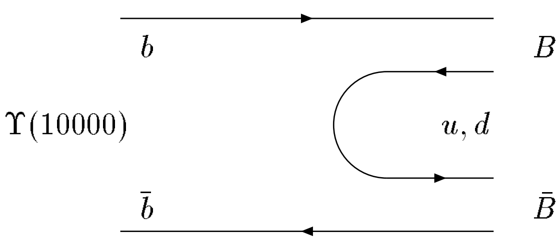
Per la bellezza delle donne
molti sono periti... Eccl.Sacra Bibbia

Because of beauty of women
many men have perished... Eccl. Bible

Victorious trend for simplicity became even more clear after another important discouvery: when in 1977 a narrow vector resonance was found at the mass around 10 GeV, $\Upsilon(1S)(9460)$,$\Gamma \approx$ 50Kev, everybody decided that there was nothing to think about, it should be just state of two new, 5th quarks of the type $\bar{s}s$ or $\bar{c}c$. This new quark was named b from *beauty* or *bottom* and was ascribed the mass around 5 GeV $\sim M_\Upsilon/2$, a half of the mass of a new meson with the hidden "beauty" $\Upsilon(1S)(9460) = (\bar{b}b)$.



Immediately searches for next excited states was begun which should similarly to $\psi(3770)$ have had essentially wider width and decay into mesons with the quantum number of "beauty". Surely it was found. It was $\Upsilon(4S)(10580)$,$\Gamma \approx$ 10 MeV. It decays almost fully to meson pair $\bar{B}B$ and these mesons B have mass $\approx 5300 = (5000 + 300)$ MeVB and mean life around 1.5 ns.



2.6 Truth or top?

Truth is not to be sought in the good fortune of any particular conjuncture of time, which is uncertain, but in the light of nature and experience, which is eternal. Francis Bacon

But it was not all the story as to this time there were already 6 leptons $e^-, \nu_e, \mu^-, \nu_\mu$, τ^-, ν_τ (and 6 antileptons) and only 5 quarks: c, u with the electric charge +2/3e and d, s, b with the electric charge -1/3e. Leaving apart theoretical foundations (though they are and serious) we see that a symmetry between quarks and leptons and between the quarks of different charge is broken. Does it mean that there exist one more, 6th quark with the new 'flavour' named "truth" or "top" ?

For 20 years its existence was taking for granted by almost all the physicists although there were also many attempts to construct models without the 6th quark. Only in 1996 t-quark was discovered at the mass close to the nucleus of $^{71}Lu^{175}$, $m_t \approx 175$GeV. We have up to now rather little to say about it and particles containing t-quark but the assertion that t-quark seems to be uncapable to form particles.

2.7 Baryons in quark model

Up to now we have considered in some details mesonic states in frameworks of unitary symmetry model and quark model up to 5 quark flavours. Let us return now to $SU(3)$ and 3-flavour quark model with quarks $q^1 = u, q^2 = d, q^3 = s$ and let us construct baryons in this model. One needs at least three quarks to form baryons so let us make a product of three 3-spinors of $SU(3)$ and search for octet in the expansion of the triple product of the IR's: $3 \times 3 \times 3 = 10 + 8 + 8' + 1$. As it could be seen it is even two octet IR's in this product so we can proceed to construct baryon octet of quarks. Expanding of the product of three 3-spinors into the sum of IR's is more complicate than for the meson case. We should symmetrize and antisymmetrize all indices to get the result:

$$q^\alpha \times q^\beta \times q^\gamma =$$

$$\frac{1}{4}(q^\alpha q^\beta q^\gamma + q^\beta q^\alpha q^\gamma + q^\alpha q^\gamma q^\beta + q^\beta q^\gamma q^\alpha + q^\gamma q^\beta q^\alpha + q^\gamma q^\alpha q^\beta)+$$

$$\frac{1}{4}(q^\alpha q^\beta q^\gamma + q^\beta q^\alpha q^\gamma - q^\alpha q^\gamma q^\beta - q^\beta q^\gamma q^\alpha - q^\gamma q^\beta q^\alpha - q^\gamma q^\alpha q^\beta)+$$

$$\frac{1}{4}(q^\alpha q^\beta q^\gamma - q^\beta q^\alpha q^\gamma + q^\alpha q^\gamma q^\beta - q^\beta q^\gamma q^\alpha + q^\gamma q^\beta q^\alpha - q^\gamma q^\alpha q^\beta)+$$

$$\frac{1}{4}(q^\alpha q^\beta q^\gamma - q^\beta q^\alpha q^\gamma - q^\alpha q^\gamma q^\beta + q^\beta q^\gamma q^\alpha - q^\gamma q^\beta q^\alpha + q^\gamma q^\alpha q^\beta) =$$

$$T^{\{\alpha\beta\gamma\}} + T^{\{\alpha\beta\}\gamma} + T^{[\alpha\beta]\gamma} + T^{[\alpha\beta\gamma]}.$$

All indices go from 1 to 3. Symmetrical tensor of the 3rd rank has the dimension $N_n^{SSS} = (n^3+3n^2+2n)/6$ and for n=3 $N_3^{SSS} = 10$. Antisymmetrical tensor of the 3rd rank has the dimension $N_n^{AAA} = (n^3-3n^2+2n)/6$ and for n=3 $N_3^{AAA} = 1$. Tensors of mixed symmetry of the dimension $N_n^{mix} = n(n^2-1)/3$ only for n=3 (N_3^{mix}=8) could be rewritten in a more suitable form as T^α_β upon using absolutely antisymmetric tensor (or Levi-Civita tensor) of the 3rd rank $\epsilon_{\beta\delta\eta}$ which transforms as the singlet IR of the group $SU(3)$. Really,

$$\epsilon_{\alpha'\beta'\gamma'} = u^\alpha_{\alpha'} u^\beta_{\beta'} u^\gamma_{\gamma'} \epsilon_{\alpha\beta\gamma} :$$

$$\epsilon_{123} = u^1_1 u^2_2 u^3_3 \epsilon_{123} + u^2_1 u^3_2 u^1_3 \epsilon_{231} + u^3_1 u^1_2 u^2_3 \epsilon_{312} + u^1_1 u^3_2 u^2_3 \epsilon_{132}$$
$$+u^3_1 u^2_2 u^1_3 \epsilon_{321} + u^2_1 u^1_2 u^3_3 \epsilon_{213} =$$

$$= (u_1^1 u_2^2 u_3^3 + u_1^2 u_2^3 u_3^1 + u_1^3 u_2^1 u_3^2 - u_1^1 u_2^3 u_3^2 - u_1^3 u_2^2 u_3^1 - u_1^2 u_2^1 u_3^3)\epsilon_{123} =$$
$$= DetU\,\epsilon_{123} = \epsilon_{123}$$

as $DetU = 1$. (The same for ϵ_{213} etc.)

For example, partly antisymmetric 8-dimensional tensor $T^{[\alpha\beta]\gamma}$ could be reduced to

$$B_\alpha^\beta|_{SU(3)}^{As} = \epsilon_{\alpha\gamma\eta} T^{[\gamma\eta]\beta},$$

and for the proton $p = B_3^1$ we would have

$$\sqrt{2}|p\rangle_{SU(3)}^{As} = \sqrt{2}B_3^1|_{SU(3)}^{As} = -|udu\rangle + |duu\rangle. \tag{2.11}$$

Instead the baryon octet based on partly symmetric 8-dimensional tensor $T^{\{\alpha\beta\}\gamma}$ could be written in terms of quark wave functions as

$$\sqrt{6}B_\beta^\alpha|_{SU(3)}^{Sy} = \epsilon_{\beta\delta\eta}\{q^\alpha, q^\delta\}q^\eta.$$

For a proton B_3^1 we have

$$\sqrt{6}|p\rangle_{SU(3)}^{Sy} = \sqrt{6}B_3^1|_{SU(3)}^{Sy} = 2|uud\rangle - |udu\rangle - |duu\rangle. \tag{2.12}$$

In order to construct fully symmetric spin-unitary spin wave function of the octet baryons in terms of quarks of definite flavour and definite spin projection we should cure only to obtain the overall functions being symmetric under permutations of quarks of all the flavours and of all the spin projections inside the given baryon. (As to the overall asymmetry of the fermion wave function we let colour degree of freedom to assure it.) Multiplying spin wave functions of Eqs.(1.19,1.21) and unitary spin wave functions of Eqs.(2.11,2.12) one gets:

$$\sqrt{18}B_\beta^\alpha|_\uparrow = \sqrt{18}(B_\beta^\alpha|_{SU(3)}^{As} \cdot t_A^j + B_\beta^\alpha|_{SU(3)}^{Sy} \cdot T_S^j). \tag{2.13}$$

Taking the proton B_3^1 as an example one has

$$\sqrt{18}|p\rangle_\uparrow = |-udu + uud\rangle \cdot |-\uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow\rangle \tag{2.14}$$
$$|2 \cdot uud - udu - duu\rangle \cdot |2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle =$$
$$= |2u_\uparrow u_\uparrow d_\downarrow - u_\uparrow d_\uparrow u_\downarrow - d_\uparrow u_\uparrow u_\downarrow$$
$$+2u_\uparrow d_\downarrow u_\uparrow - u_\uparrow u_\downarrow d_\uparrow - d_\uparrow u_\downarrow u_\uparrow + 2d_\downarrow u_\uparrow u_\uparrow - u_\downarrow u_\uparrow d_\uparrow - u_\downarrow d_\uparrow u_\uparrow\rangle.$$

We have used here that

$$|uud\rangle \cdot |\downarrow\uparrow\uparrow\rangle = |u_\downarrow u_\uparrow d_\uparrow\rangle$$

and so on.

Important note. One could safely use instead of the Eq.(2.14) the shorter version but where one already cannot change the order of spinors at all!

$$\sqrt{6}|p\rangle = \sqrt{6}|B_3^1\rangle_\uparrow = |2u_\uparrow u_\uparrow d_\downarrow - u_\uparrow d_\uparrow u_\downarrow - d_\uparrow u_\uparrow u_\downarrow\rangle. \tag{2.15}$$

The wave function of the isosinglet Λ has another structure as one can see oneself upon calcolating

$$2|\Lambda\rangle_\uparrow = -\sqrt{6}|B_3^3\rangle_\uparrow = |d_\uparrow s_\uparrow u_\downarrow + s_\uparrow d_\uparrow u_\downarrow - u_\uparrow s_\uparrow d_\downarrow - s_\uparrow u_\uparrow d_\downarrow\rangle. \tag{2.16}$$

Instead decuplet of baryon resonances $T^{\{\alpha\beta\gamma\}}$ with $J^P = \frac{3}{2}^+$ would be written in the form of a so called weight diagram (it gives a convenient grafic image of the $SU(3)$ IR's on the 2-parameter plane which is characterized by basic elements, in our case the 3rd projection of isospin I_3 as an absciss and hypercharge Y as a ordinate) which for decuplet has the form of a triangle:

$$\begin{array}{ccccccc} \Delta^- & & \Delta^0 & & \Delta^+ & & \Delta^{++} \\ & \Sigma^{*-} & & \Sigma^{*0} & & \Sigma^{*+} & \\ & & \Xi^{*-} & & \Xi^{*0} & & \\ & & & \Omega^- & & & \end{array}$$

and has the following quark content:

$$\begin{array}{ccccccc} ddd & & udd & & uud & & uuu \\ & sdd & & sud & & suu & \\ & & ssd & & ssu & & \\ & & & sss & & & \end{array}$$

In the $SU(3)$ group all the weight diagram are either hexagones or triangles and often are convevient in applications.

For the octet the weight diagram is a hexagone with the 2 elements in center:

$$\begin{array}{ccc} & n \quad p & \\ \Sigma^- & (\Sigma^0, \Lambda) & \Sigma^+ \\ & \Xi^- \quad \Xi^0 & \end{array}$$

or in terms of quark content:

$$\begin{array}{ccc} & udd \quad uud & \\ sdd & sud & suu \\ & ssd \quad ssu & \end{array}$$

The discouvery of 'charm' put a problem of searching of charm baryons. And indeed they were found! Now we already know $\Lambda_c^+(2285, 1 \pm 0, 6 M\text{э}B)$, $\Lambda_c^+(2625, 6{\pm}0, 8 M\text{э}B)$, $\Sigma_c^{++,+,0}(2455)$, $\Xi_c^{+,0}(2465)$. Let us try to classify them along the IR's of groups $SU(4)$ and $SU(3)$. Now we make a product of three 4-spinors $q^\alpha, \alpha = 1,2,3,4$. Tensor structure is the same as for $SU(3)$. But dimensions of the IR's are certainly others: $4 \times 4 \times 4 = 20_4 + 20'_4 + 20'_4 + \bar{4}$. A symmetrical tensor of the 3rd rank of the dimension $N_n^{SSS} = (n^3 + 3n^2 + 2n)/6$ in $SU(4)$ has the dimension 20 and is denoted usually as 20_4. Reduction of the IR 20_4 in IR's of the group $SU(3)$ has the form $20_4 = 10_3 + 6_3 + 3_3 + 1_3$.

There is an easy way to obtain the reduction in terms of the corresponding dimensions. Really, as it is almost obvious, 4- spinor of $SU(4)$ *reduces to* $SU(3)$ *IR's as* $4_4 = 3_3 + 1_3$. *So the product Eqs.(1.29,1.30) for* $n = 4$ $4_4 \times 4_4 = 10_4 + 6_4$ *would reduce as*

$$(3_3 + 1_3) \times (3_3 + 1_3) = 3_3 \times 3_3 + 3_3 \times 1_3 + 1_3 \times 3_3 + 1_3 \times 1_3 =$$

$$6_3 + \bar{3}_3 + 3_3 + 3_3 + 1_1.$$

As antisymmetric tensor of the 2nd rank $T^{[\alpha\beta]}$ *of the dimension* $N_n^{AA} = n(n-1)/2$ *equal to 6 at n=4 and denoted by us as* 6_4 *should be equal to its conjugate* $T_{[\alpha\beta]}$ *due to the absolutely anisymmetric tensor* $\epsilon_{\alpha\beta\gamma\delta}$ *and so it has the following* $SU(3)$ *content:* $6_4 = 3_3 + \bar{3}_3$. *The remaining symmetric tensor of the 2nd rank* $T^{\{\alpha\beta\}}$ *of the dimension* $N_n^{SS} = n(n+1)/2$ *equal to 10 at n=4 and denoted by us as* 10_4 *would have then* $SU(3)$ *content* $10_4 = 6_3 + 3_3 + 1_3$. *The next step would be to construct reduction to* $SU(3)$ *for products* $6_4 \times 4_4 =$

$20'_4 + \bar{4}_4$ *(see Eq.(1.18) at* $n = 4$*) and* $10_4 \times 4_4 = 20'_4 + 20_4$ *(Eq.(1.20) at* $n = 4$*). Their sum would give us the answer for* $4 \times 4 \times 4$.

$$6_4 \times 4_4 = (3_3 + \bar{3}_3) \times (3_3 + 1_3) = \bar{4}_4 + 20'_4 =$$

$$= 3_3 \times 3_3 + \bar{3}_3 \times 3_3 + 3_3 + \bar{3}_3 =$$

$$= (\bar{3}_3 + 1_3) + (8_3 + 6_3 + \bar{3}_3 + 3_3.)$$

As $20'_4$ *is now known it easy to obtain* 20_4*:*

$$20_4 = 10_4 \times 4_4 - 20'_4 = 10_3 + 6_3 + 3_3 + 1_3.$$

Tensor calculus with $q^\alpha = \delta^\alpha_a q^a + \delta^\alpha_4 q^4$ *would give the same results.*

So 20_4 contains as a part 10-plet of baryon resonances $3/2^+$. As to the charm baryons $3/2^+$ there are two candidate: $\Sigma_c(2520)$ and $\Xi_c(2645)$ (J^P has not been measured; $3/2^+$ is the quark model prediction) Note by the way that J^P of the Ω^- which manifested triumph of $SU(3)$ has not been measured since 1964 that is for more than 40 years! Nevertheless everybody takes for granted that its spin-parity is $3/2^+$.

Instead baryons $1/2^+$ enter $20'$-plet described by traceless tensor of the 3rd rank of mixed symmetry $B^\alpha_{[\gamma\beta]}$ antisymmetric in two indices in square brackets:

$$\sqrt{6} B^\alpha_{[\gamma\beta]} = \epsilon_{\gamma\beta\delta\eta}\{q^\alpha, q^\delta\} q^\eta$$

$\alpha, \beta, \gamma, \delta, \eta = 1, 2, 3, 4$. This $20'$-plet as we have shown is reduced to the sum of the $SU(3)$ IR's as $20'_4 = 8_3 + 6_3 + 3_3 + \bar{3}_3$. It is convenient to choose reduction along the multiplets with the definite value of charm. Into 8-plet with $C = 0$ the usual baryon octet of the quarks u,d,s (2.1) is naturally placed. The triplet should contain not yet discouvered in a definite way doubly-charmed baryons:

$$\begin{array}{cc} \Xi^+_{cc} & \Xi^{++}_{cc} \\ \multicolumn{2}{c}{\Omega^+_{cc}} \end{array}$$

with the quark content

$$\begin{array}{cc} ccd & ccu \\ \multicolumn{2}{c}{ccs,} \end{array}$$

and, for example, the wave function of Ξ^+_{cc} in terms of quarks reads

$$\sqrt{6}|\Xi^+_{cc}\rangle_\uparrow = |2c_\uparrow c_\uparrow u_\downarrow - c_\uparrow u \uparrow c_\downarrow - u_\uparrow c_\uparrow c_\downarrow\rangle. \qquad (2.17)$$

Antitriplet contains already discouvered baryons with $C = 1$

$$\Lambda_c^+$$

$$\Xi_c^0 \quad \Xi_c^+$$

with the quark content

$$udc$$

$$dsc \quad usc,$$

and, for example, the wave function of Ξ_c^+ in terms of quarks reads

$$2|\Xi_c^+\rangle_\uparrow = |s_\uparrow c_\uparrow u_\downarrow + c_\uparrow s_\uparrow u_\downarrow - u_\uparrow c_\uparrow s_\downarrow - c_\uparrow u_\uparrow s_\downarrow\rangle. \tag{2.18}$$

Sextet contains discouvered baryons with $C = 1$

$$\Sigma_c^0 \quad \Sigma_c^+ \quad \Sigma_c^{++}$$

$$\Xi_c^{\prime 0} \quad \Xi_c^{\prime +}$$

$$\Omega_c^0$$

(Note that their quantum numbers are not yet measured!) with the quark content

$$ddc \quad udc \quad uuc$$

$$dsc \quad usc$$

$$ssc$$

and, for example, the wave function of $\Xi_c^{\prime +}$ in terms of quarks reads

$$\sqrt{12}|\Xi_c^{\prime +}\rangle_\uparrow = |2u_\uparrow s_\uparrow c_\downarrow + 2c_\uparrow u_\uparrow d_\downarrow \tag{2.19}$$

$$-u_\uparrow c_\uparrow s_\downarrow - c_\uparrow u_\uparrow s_\downarrow - s_\uparrow c_\uparrow u_\downarrow - c_\uparrow s_\uparrow u_\downarrow\rangle.$$

Note also that in the $SU(4)$ group absolutely antisymmetric tensor of the 4th rank $\epsilon_{\gamma\beta\delta\eta}, \gamma, \beta, \delta, \eta$, transforms as singlet IR and because of that antisymmetric tensor of the 3th rank in $SU(4)$ $T^{[\alpha\beta\gamma]}, \alpha, \beta, \gamma = 1, 2, 3, 4$, transforms not as a singlet IR as in $SU(3)$ (what is proved in $SU(3)$ by reduction with the tensor $\epsilon_{\beta\delta\eta}, \beta, \delta, \eta = 1, 2, 3$) but instead along the conjugated spinor representation $\bar{4}$ (which also can be proved by the reduction of it with the tensor $\epsilon_{\gamma\beta\delta\eta}, \gamma, \beta, \delta, \eta = 1, 2, 3, 4$).

We return here to the problem of reduction of the IR of some group into the IR's of minor group or, in particular, to IR's of a production of two minor groups. Well-known example is given by the group $SU(6) \supset SU(3) \times SU(2)_S$ which in nonrelativistic case has unified unitary model group $SU(3)$ and spin group $SU(2)_S$. In the framework of $SU(6)$ quarks belong to the spinor of dimension 6_6 which in the space of $SU(3) \times SU(2)_S$ could be written as $6_6 = (3, 2)$ where the first symbol in brackets means 3-spinor of $SU(3)$ while the 2nd symbol just states for 2-dimensional spinor of $SU(2)$. Let us try now to form a product of 6-spinor and corresponding 6-antispinor and reduce it to IR's of the product $SU(3) \times SU(2)_S$:

$$\bar{6}_6 \times 6_6 = 35_6 + 1_6 = (\bar{3}, 2) \times (3, 2) = (\bar{3} \times 3, 2 \times 2) =$$

$$= (8 + 1, 3 + 1) = [(8, 1) + (8, 3) + (1, 3)] + (1, 1),$$

that is, in 35_6 there are exactly eight mesons of spin zero and 9=8+1 vector mesons, while there is also zero spin meson as a $SU(6)$ singlet. This result suits nicely experimental observations for light (of quarks u, d, s) mesons. In order to proceed further we form product of two 6-spinors first according to our formulae, just dividing the product into symmetric and antisymmetric IR's of the rank 2:

$$6_6 \times 6_6 = 21_6 + 15_6 = (3, 2) \times (3, 2) = (3 \times 3, 2 \times 2) =$$

$$= (6_3 + \bar{3}_3, 3 + 1) =$$

$$\{(6_3, 3) + (3_3, 1)\}_{21} + [(6_3, 1) + (\bar{3}_3, 3)]_{15},$$

dimensions of symmetric tensors of the 2nd rank being $n(n + 1)/2$ while of those antisymmetric $n(n - 1)/2$. Now we go to product $15_6 \times 6_6$ which should results as already we have seen in the sum of two IR's of the 3rd rank , one of them being antisymmetric of the 3rd rank ($N^{AAA} = n(n^2 - 3n + 2)/6$) and the other being of mixed symmetry ($N_{mix} = n(n^2 - 1)/3$):

$$15_6 \times 6_6 = 20_6 + 70_6 = [(6_3, 1) + (\bar{3}_3, 3)] \times (3, 2) =$$

$$= (6 \times 3, 2) + (\bar{3} \times 3, 3 \times 2) =$$

$$= [(8, 2) + (1, 4)]_{20} + [(8, 4) + (10, 2) + (8, 2) + (1, 2)]_{70}.$$

Instead the product $21_6 \times 6_6$ should result as already we have seen into the sum of two IR's of the 3rd rank , one of them being symmetric of the 3rd *rank* ($N^{SSS} = n(n^2+3n+2)/6$) *and the other being again of mixed symmetry* ($N_{mix} = n(n^2-1)/3$) *and we used previous result to extract the reduction of the* 56_6*-plet:*

$$21_6 \times 6_6 = 56_6 + 70_6 = \{(6_3, 3) + (\bar{3}_3, 1)\} \times (3, 2) =$$

$$= (6_3 \times 3_3, 3 \times 2) + (\bar{3}_3 \times 3, 1 \times 2) =$$

$$= \{((8, 2) + (10, 4))\}_{56} + [(8, 2) + (1, 4)]_{20} +$$

$$+[(8, 4) + (10, 2) + (8, 2) + (1, 2)]_{70}.$$

We now see eminent result of $SU(6)$ *that is that in one IR* 56_6 *there are octet of baryons of spin 1/2 and decuplet of baryonic resonances of spin 3/2! Note that quark model with all the masses, magnetons etc equal just reproduces* $SU(6)$ *model as it should be. In some sense 3-quark model gives the possibility of calculations alternative to tensor calculus of* $SU(6)$ *group.*

Some words also on reduction of IR of some group to IR's of the sum of minor groups. We have seen an example of reduction of the IR of $SU(4)$ *into those of* $SU(3)$*. For future purposes let us consider some examples of the reduction of the IR's of* $SU(5)$ *into those of the direct sum* $SU(3)$*+*$SU(2)$*. Here we just write 5-spinor of* $SU(5)$ *as a direct sum:* $5_5 = (3_3, 1) + (1_3, 2)$*. Forming the product of two 5-spinors we get:*

$$5_5 \times 5_5 = 15_5 + 10_5 = (3_3, 1) + (1_3, 2) \times (3_3, 1) + (1_3, 2) =$$

$$= (3_3 \times 3_3, 1) + (3_3, 2) + (3_3, 2) + (1_3, 2 \times 2) =$$

$$\{(6, 1) + (3_3, 2) + (1, 3)\}_{15} + [(3_3, 2) + (\bar{3}_3, 1) + (1, 1)]_{10},$$

that is, important for $SU(5)$ *group IR's of dimensions 5 and 10 have the following reduction to the sum* $SU(3)$*+*$SU(2)$*:*

$$5_5 = (3_3, 1) + (1_3, 2),$$

$$10_5 = (3_3, 2) + (\bar{3}_3, 1) + (1, 1).$$

In this case it is rather easy an exercise to proceed also with tensor calculus.

Chapter 3

Currents in unitary symmetry and quark models

3.1 Electromagnetic current in the models of unitary symmetry and of quarks

3.1.1 On magnetic moments of baryons

Main properties of electromagnetic interaction are assumed to be known.

Electromagnetic current of baryons as well as of quarks can be written in a similar way to electrons in the theory with the Dirac equation, only we should account in some way for the non-point-like structure of baryons introducing one more Lorentz structure and two form factors. This current can be deduced from the interaction Lagrangian of the baryon with the electric charge e and described by a spinor $u_B(p)$ and electromagnetic field $A^\mu(x)$ characterized by its polarization vector ϵ^μ:

$$\frac{e}{2M_B(1-\frac{q^2}{4M_B^2})}\bar{u}_B(p_2)[P_\mu G_E(q^2)-$$

$$-i\epsilon_{\mu\nu\rho\sigma}P^\nu q^\rho \gamma^\sigma \gamma_5 G_M(q^2)]u_B(p_1)\epsilon^\mu =$$

$$(\hat{p}-m_B)u_B=0, \qquad q=p_1-p_2, \qquad P=p_1+p_2$$

$$2i\sigma_{\mu\nu}=[\gamma_\mu,\gamma_\nu],$$

G_E being electric form factor, $G_E(0) = 1$,while G_M is a magnetic form factor and $G_M(0) = \mu_B$ is a total magnetic moment of the baryon in terms of proper magnetons $eh/2m_Bc$. Transition to the model of unitary symmetry means that instead of the spinor u_B written for every baryon one should now put the whole octet B^α_β.

What are properties of electromagnetic current in the unitary symmetry? Let us once more remind Gell-Mann–Nishijima relation between the particle charge Q, 3rd component of the isospin I_3 and hypercharge Y,

$$Q = I_3 + \frac{1}{2}Y.$$

As Q is just the integral over 4th component of electromagnetic current, it means that the electromagnetic current is just a superposition of the 3rd component of isovector current and of the hypercharge current which is isoscalar.

So it can be related to the component $J^1_{\mu 1}$ of the octet of vector currents $J^\alpha_{\mu\beta}$. (More or less in the same way as mass breaking was described by the 33 component of the baryonic current but without specifying its space-time properties.) The part of the current related to the electric charge should assure right values of the baryon charges. Omitting for the moment space-time indices we write

$$eJ^1_1 = e(\bar{B}^\alpha_1 B^1_\alpha - \bar{B}^1_\alpha B^\alpha_1).$$

Here $p = B^1_3$ and so on, are octet baryons with $J^P = \frac{1}{2}^+$. One can see that all the charges of baryons are reproduced .

But the part treating magnetic moments should not coincide in form with that for their charges, as there are anomal magnetic moments in addition to those normal ones. The total magnetic moment is a sum of these two magnetic moments for charged baryons and just equal to anomal one for the neutral baryons.

While constructing baryon current suitable for description of the magnetic moments as a product of baryon and antibaryon octets we use the fact that there are possible as we already know two different tensor structures (which reflects existence of two octets in the expansion $8\times 8 = 1+8+8+10+10^*+27$).

$$J^\alpha_\beta = F(\bar{B}^\gamma_\beta B^\alpha_\gamma - \bar{B}^\alpha_\gamma B^\gamma_\beta) + D(\bar{B}^\gamma_\beta B^\alpha_\gamma + \bar{B}^\alpha_\gamma B^\gamma_\beta) - \frac{2}{3}\delta^\alpha_\beta D\bar{B}^\gamma_\eta B^\eta_\gamma,$$

and trace of the current should be zero, $J_\gamma^\gamma = 0, \alpha, \beta, \gamma, \eta = 1, 2, 3,.$ Then electromagnetic current related to magnetic moments (we omit space-time indices here) will have the form

$$J_1^1 = F(\bar{B}_1^\alpha B_\alpha^1 - \bar{B}_\alpha^1 B_1^\alpha) + D(\bar{B}_1^\alpha B_\alpha^1 + \bar{B}_\alpha^1 B_1^\alpha - \frac{2}{3}\bar{B}_\beta^\alpha B_\alpha^\beta),$$

wherefrom magnetic moments of the octet baryons read:

$$\begin{array}{ll} \mu(p) = F + \frac{1}{3}D, & \mu(\Sigma^+) = F + \frac{1}{3}D, \\ \mu(n) = -\frac{2}{3}D, & \mu(\Sigma^-) = -F + \frac{1}{3}D, \\ \mu(\Xi^0) = -\frac{2}{3}D, & \mu(\Sigma^0) = \frac{1}{3}D, \\ \mu(\Xi^-) = -F + \frac{1}{3}D & \mu(\Lambda^0) = -\frac{1}{3}D \end{array} \tag{3.1}$$

(Remind that here $B_3^1 = p$.) Agreement with experiment in terms of only F and D constants proves to be rather poor. But many modern model developed for description of the baryon magnetic moments contain these contributions as leading ones to which there are added minor corrections often in the frameworks of very exquisite theories.

Let us put here experimental values of the measured magnetic moments in nucleon magnetons.

$$\begin{array}{ll} \mu(p) = 2.793 & \mu(\Sigma^+) = 2,458 \pm 0.010, \\ \mu(n) = -1.913, & \mu(\Sigma^-) = -1.16 \pm 0.025, \\ & \mu(\Xi^0) = -1.250 \pm 0.014, \\ \mu(\Xi^-) = -0.6507 \pm 0.0025 & \mu(\Lambda^0) = -0.613 \pm 0.04 \end{array} \tag{3.2}$$

And in what way could we construct electromagnetic current of quarks? It is readily written from Dirac electron current:

$$J_\mu^{el-m} = \frac{2}{3}\bar{t}\gamma_\mu t + \frac{2}{3}\bar{c}\gamma_\mu c + [\frac{2}{3}\bar{u}\gamma_\mu u -$$

$$-\frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s] - \frac{1}{3}\bar{b}\gamma_\mu b,$$

and we have put in square brackets electromagnetic current of the 3-flavor model.

And how could we resolve problem of the baryon magnetic moments in the framework of the quark model?

For this purpose we need explicit form of the baryon wave functions with the given 3rd projection of the spin in terms of quark wave functions also with definite 3rd projections of the spin. These wave functions have been given in previous lectures. We assume that in the nonrelativistic limit magnetic moment of the baryon would be a sum of magnetic moments of quarks, while operator of the magnetic moment of the quark q would be $\mu_q \sigma_z^q$ (quark on which acts operator of the magnetic moment is denoted by *). Magnetic moment of proton is obtained as (here $q_1 = q_\uparrow, q_2 = q_\downarrow, q = u, d, s.$)

$$\mu_p = \sum\nolimits_{q=u,d} < p_\uparrow | \hat{\mu}_q \sigma_z^q | p_\uparrow >=$$

$$\frac{1}{6} < 2u_1u_1d_2 - u_1d_1u_2 - d_1u_1u_2 | \hat{\mu}_q \sigma_z^q | 2u_1u_1d_2 - u_1d_1u_2 - d_1u_1u_2 >=$$

$$\frac{1}{6} \sum\nolimits_{q=u,d} < 2u_1u_1d_2 - u_1d_1u_2 - d_1u_1u_2 | \hat{\mu}_q | 2u_1^*u_1d_2 + 2u_1u_1^*d_2 -$$

$$-2u_1u_1d_2^* - u_1^*d_1u_2 - u_1d_1^*u_2 + u_1d_1u_2^* -$$

$$-d_1^*u_1u_2 - d_1u_1^*u_2 + d_1u_1u_2^* >=$$

$$\frac{1}{6}(4\mu_u + 4\mu_u - 4\mu_d + \mu_u + \mu_d - \mu_u + \mu_d + \mu_u - \mu_u) =$$

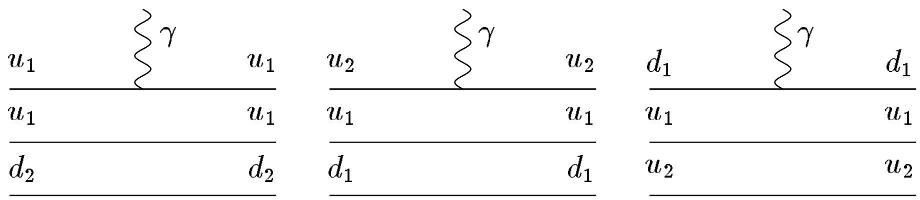
$$\frac{1}{6}(8\mu_u - 2\mu_d) = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d,$$

where we have used an assumption that two of three quarks are always spectators so that

$$< u_1u_1d_2 | \hat{\mu}_q | u_1^*u_1d_2 >=$$

$$=< u_1 | \hat{\mu}_q | u_1^* >= \mu_u \quad etc.$$

Corresponding quark diagrams could be written as (we put only some of them, the rest could be written in the straightforward manner):



In a similar way we can calculate in NRQM magnetic moment of neutron:

$$\mu_n = \sum\nolimits_{q=u,d} < n_\uparrow | \mu_q \sigma_z^q | n_\uparrow >=$$

$$\frac{1}{6} < 2d_1d_1u_2 - d_1u_1d_2 - u_1d_1d_2 | \mu_q \sigma_z^q | 2d_1d_1u_2 - d_1u_1d_2 - u_1d_1d_2 >=$$

$$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u,$$

magnetic moment of Σ^+:

$$\mu(\Sigma^+) = \sum\nolimits_{q=u,s} < \Sigma_\uparrow^+ | \mu_q \sigma_z^q | \Sigma_\uparrow^+ >=$$

$$\frac{1}{6} < 2u_1u_1s_2 - u_1s_1u_2 - s_1u_1u_2 | \mu_q \sigma_z^q | 2u_1u_1s_2 - u_1s_1u_2 - s_1u_1u_2 >=$$

$$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s,$$

magnetic moment of Σ^- :

$$\mu(\Sigma^-) = \sum\nolimits_{q=d,s} < \Sigma_\uparrow^- | \mu_q \sigma_z^q | \Sigma_\uparrow^- >=$$

$$\frac{1}{6} < 2d_1d_1s_2 - d_1s_1d_2 - s_1d_1d_2 | \mu_q \sigma_z^q | 2d_1d_1s_2 - d_1s_1d_2 - s_1d_1d_2 >=$$

$$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s,$$

magnetic moment of Ξ^0 :

$$\mu(\Xi^0) = \sum\nolimits_{q=u,s} < \Xi_\uparrow^0 | \mu_q \sigma_z^q | \Xi_\uparrow^0 >=$$

$$\frac{1}{6} < 2s_1s_1u_2 - s_1u_1s_2 - u_1s_1s_2|\mu_q\sigma_z^q|2s_1s_1u_2 - s_1u_1s_2 - u_1s_1s_2 >=$$

$$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u,$$

magnetic moment of Ξ^- :

$$\mu(\Xi^-) = \sum\nolimits_{q=d,s} < \Xi^-_\uparrow|\mu_q\sigma_z^q|\Xi^-_\uparrow >=$$

$$\frac{1}{6} < 2s_1s_1d_2 - s_1d_1s_2 - d_1s_1s_2|\mu_q\sigma_z^q|2s_1s_1d_2 - s_1d_1s_2 - d_1s_1s_2 >=$$

$$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d,$$

and finally magnetic moment of Λ (which we write in some details as it has the wave function of another type):

$$\mu_\Lambda = \sum\nolimits_{q=u,d,s} < \Lambda_\uparrow|\mu_q\sigma_z^q|\Lambda_\uparrow >=$$

$$\frac{1}{4} < u_1s_1d_2 + s_1u_1d_2 - d_1s_1u_2 - s_1d_1u_2|\mu_q\sigma_z^q|$$

$$|u_1s_1d_2 + s_1u_1d_2 - d_1s_1u_2 - s_1d_1u_2 >=$$

$$\frac{1}{4} < u_1s_1d_2 + s_1u_1d_2 - d_1s_1u_2 - s_1d_1u_2|\mu_q|u_1^*s_1d_2 + u_1s_1^*d_2 - u_1s_1d_2^* +$$

$$+s_1^*u_1d_2 + s_1u_1^*d_2 - s_1u_1d_2^*$$

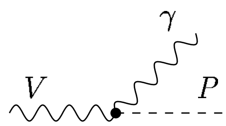
$$-d_1^*s_1u_2 - d_1s_1^*u_2 + d_1s_1u_2^* - s_1^*d_1u_2 - s_1d_1^*u_2 + s_1d_1u_2^* >=$$

$$\frac{1}{4}(\mu_u + \mu_s - \mu_d + \mu_s + \mu_u - \mu_d + \mu_d + \mu_s - \mu_u + \mu_s + \mu_d - \mu_u) = \mu_s,$$

In terms of these three quark magnetons it is possible to adjust magnetic moments of baryons something like up to 10% accuracy.

3.1.2 Radiative decays of vector mesons

Let us look now at radiative decays of vector meson $V \to P + \gamma$.



Let us write in the framework of unitary symmetry model an electromagnetic current describing radiative decays of vector mesons as the 11-component of the octet made from the product of the octet of preusoscalar mesons and nonet of vector mesons:

$$J_1^1 = P_\gamma^1 V_1^\gamma + P_1^\gamma V_\gamma^1 - \frac{2}{3} SpPV =$$

$$2P_1^1 V_1^1 + (P_2^1 V_1^2 + P_3^1 V_1^3) + (P_1^2 V_2^1 + P_1^3 V_3^1) - \frac{2}{3} SpPV =$$

$$2(\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{6}\eta)(\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega) - \frac{2}{3}(\pi^0)\rho^0 + \pi^+\rho^- + \pi^-)\rho^+) + ...$$

We take interaction Lagrangian describing these transitions in the form

$$L = g_{V\to P\gamma} J_{1\mu}^1 A_\mu.$$

Performing product of two matrix and extracting 11 component we obtain for amplitudes of radiative decays in unitary symmetry:

$$M(\rho^0) \to \pi^0\gamma) = (2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{2}{3})g_{V\to P\gamma} = \frac{1}{3} g_{V\to P\gamma},$$

$$M(\rho^\pm \to \pi^\pm\gamma) = (1 - \frac{2}{3})g_{V\to P\gamma} = \frac{1}{3} g_{V\to P\gamma},$$

$$M(\omega^0 \to \pi^0\gamma) = 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} g_{V\to P\gamma} = g_{V\to P\gamma} =$$

As masses of ρ and ω^0 mesons are close to each other we neglect a difference in phase space and obtain the following widths of the radiative decays:

$$\Gamma(\omega^0 \to \pi^0\gamma) : \Gamma(\rho^\pm \to \pi^\pm\gamma) = 9 : 1,$$

while experiment yields:

$$(720 \pm 50)Kev : (120 \pm 30)Kev$$

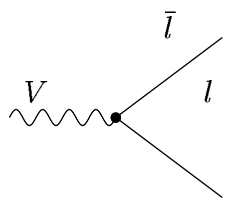
Radiative decay of ϕ meson proves to be prohibited in unitary symmetry,

$$\Gamma(\phi \to \pi^0 \gamma) = 0.$$

The experiment shows very strong suppression of this decay, $(6 \pm 0,6)Kev$.

3.1.3 Leptonic decays of vector mesons $V \to l^+ l^-$

Let us now construct in the framework of unitary symmetry model an electromagnetic current describing leptonic decays of vector mesons $V \to l^+ l^-$.



In sight of previous discussion it is easily to see that it would be sufficient to extract an octet in the nonet of vector mesons and then take 11-component:

$$J_1^1 = g_{V\bar{l}l}(V_1^1 - \frac{1}{3}V_\alpha^\alpha) = g_{V\bar{l}l}[(\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega) - \frac{1}{3}(\sqrt{2}\omega + \phi)] =$$

$$= g_{V\bar{l}l}(\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{3\sqrt{2}}\omega - \frac{1}{3}\phi)$$

Ratio of leptonic widths is predicted in unitary symmetry as

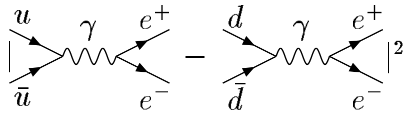
$$\Gamma(\rho^0 \to e^+e^-) : \Gamma(\omega^0 \to e^+e^-) : \Gamma(\phi^0 \to e^+e^-) = 9 \quad : \quad 1 : \quad 2,$$

which agrees well with the experimental data:

$$6,8Kev \quad : 0,6Kev \quad : 1,3Kev \quad .$$

Let us perform calculations of these leptonic decays in quark model upon using quark wave functions of vector mesons For $\rho^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$

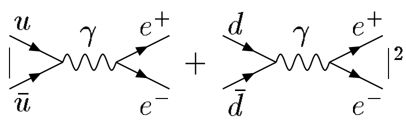
$$\Gamma(\rho^0 \to e^+ + e^-) =$$



$$= \frac{\kappa}{2}[\tfrac{2}{3} - (-\tfrac{1}{3})]^2 = \frac{\kappa}{2}.$$

For $\omega^0 = \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$

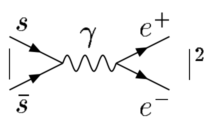
$$\Gamma(\omega^0 \to e^+ + e^-) =$$



$$= \frac{\kappa}{2}[\tfrac{2}{3} + (-\tfrac{1}{3})]^2 = \frac{\kappa}{18}.$$

For $\phi = (\bar{s}s)$

$$\Gamma(\phi^0 \to e^+ + e^-) =$$



$$= \kappa(\tfrac{1}{3})^2 = \frac{\kappa}{2}\frac{2}{9}.$$

It is seen that predictions of unitary symmetry model of the quark model coincide with each other and agree with the experimental data.

3.2 Photon as a gauge field

Up to now we have treated photon at the same level as other particles that as a boson of spin 1 and mass zero. But it turns out that existence of the photon can be thought as effect of local gauge invariance of Lagrangian describing free field of a charged fermion of spin 1/2, let it be electron. Free motion of electron is ruled by Dirac equation $(\partial_\mu\gamma_\mu - m_e)\psi_e(x) = 0$ which could be obtained from Lagrangian

$$L_0 = \bar{\psi}_e(x)\partial_\mu\gamma_\mu\psi_e(x) + m_e\bar{\psi}_e(x)\psi_e(x).$$

This Lagrangian is invariant under gauge transformation

$$\psi_e'(x) = e^{i\alpha}\psi_e(x),$$

α being arbitrary real phase. Let us demand invariance of this Lagrangian under similar but local transformation, that is when α is a function of x:

$$\psi_e'(x) = e^{i\alpha(x)}\psi_e(x),$$

It is easily seen that L_0 is not invariant under such local gauge transformation:

$$L_0' = \bar{\psi}_e'(x)\partial_\mu\gamma_\mu\psi_e'(x) + m_e\bar{\psi}_e'(x)\psi_e'(x) =$$

$$\bar{\psi}_e(x)\partial_\mu\gamma_\mu\psi_e(x) + i\frac{\partial\alpha(x)}{\partial x_\mu}\bar{\psi}_e(x)\gamma_\mu\psi_e(x) - m_e\bar{\psi}_e(x)\psi_e(x).$$

In order to cancel the term violating gauge invariance let us introduce some vector field A_μ with its own gauge transformation

$$A_\mu' = A_\mu - \frac{1}{e}\frac{\partial\alpha(x)}{\partial x_\mu},$$

and introduce also an interaction of it with electron through Lagrangian

$$ie\bar{\psi}_e(x)\gamma_\mu\psi_e(x)A_\mu,$$

e being coupling constant (or constant of interaction). But we cannot introduce a mass of this field as obviously mass term of a boson field in Lagrangian $m_\gamma A_\mu A^\mu$ is not invariant under chosen gauge transformation for the vector field A_μ. Let us now identify the field A_μ with the electromagnetic field and write the final expression of the Lagrangian invariant under local gauge transformations of the Abelian group $U(1)$

$$L_0 = \bar\psi_e(x)\partial_\mu\gamma_\mu\psi_e(x) + ie\bar\psi_e(x)\gamma_\mu\psi_e(x)A_\mu$$
$$+m_e\bar\psi_e(x)\psi_e(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu}$ describe free electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

satisfying Maxwell equations

$$\partial_\mu F_{\mu\nu} = 0, \quad \partial_\mu A_\mu = 0.$$

The 4-vector potential of the electromagnetic field $A_\mu = (\phi, \quad \vec{A})$ is related to measured on experiment magnetic $\vec{H}$ and electric fields $\vec{E}$ by the relations

$$\vec{H} = rot\vec{A}, \quad \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - grad \quad \phi,$$

while tensor of the electromagnetic field $F_{\mu\nu}$ is written in terms of the fields $\vec{E}$ and $\vec{H}$ as

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu = \begin{pmatrix} 0 & E_x & E_x & E_x \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}.$$

Maxwell equations (in vacuum) in the presence of charges and currents read

$$rot\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}, \quad div\vec{H} = 0,$$
$$rot\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}, \quad div\vec{E} = 4\pi\rho,$$

where ρ is the density of electrical charge and j is the electric current density. In the 4-dimensional formalism Maxwell equations (in vacuum) in the presence of charges and currents can be written as

$$\partial_\mu F_{\mu\nu} = j_\nu, \qquad j_\nu = (\rho, \quad \vec{j}),$$
$$\epsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0, \quad \partial_\mu A_\mu = 0.$$

3.3 ρ meson as a gauge field

In 1954 that is more than half-century ago Yang and Mills decided to try to obtain also ρ meson as a gauge field. The ρ mesons were only recently discouvered and seemed to serve ideally as quants of strong interaction.

Similar to photon case let us consider Lagrangian of the free nucleon field where nucleon is just isospinor of the group $SU(2)_I$ of isotopic transformations with two components, that is proton (chosen as a state with I_3=+1/2) and neutron (chosen as a state with I_3=-1/2):

$$L_0 = \bar{\psi}_N(x)\partial_\mu\gamma_\mu\psi_N(x) + m_N\bar{\psi}_N(x)\psi_N(x).$$

This Lagrangian is invariant under global gauge transformations in isotopic space

$$\psi'_N(x) = e^{i\vec{\alpha}\vec{\tau}}\psi_N(x),$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are three arbitrary real phases. Let us demand now invariance of the Lagrangian under similar but local gauge transformation in isotopic space when $\vec{\alpha}$ is a function of x:

$$\psi'_N(x) = e^{i\vec{\alpha}(x)\vec{\tau}}\psi_N(x).$$

However as in the previous case The L_0 is not invariant under such local gauge transformation:

$$L'_0 = \bar{\psi}'_N(x)\partial_\mu\gamma_\mu\psi'_N(x) + m_N\bar{\psi}'_N(x)\psi'_N(x) = \bar{\psi}_N(x)\partial_{x_\mu}\gamma_\mu\psi_N(x) -$$

$$+ i\bar{\psi}_N(x)\gamma_\mu\frac{\partial\vec{\tau}\vec{\alpha}(x)}{\partial x_\mu}\psi_N(x) + m_N\bar{\psi}_N(x)\psi_N(x).$$

In order to cancel term breaking gauge invariance let us introduce an isotriplet of vector fields $\vec{\rho}_\mu$ with the gauge transformation

$$\vec{\tau}\vec{\rho}'_\mu = U\vec{\tau}\vec{\rho}_\mu U^\dagger - \frac{1}{g_{NN\rho}}\frac{\partial U}{\partial x_\mu}U^\dagger$$

where $U = e^{i\vec{\alpha}(x)\vec{\tau}}$. An interaction of this isovector vector field could be given by Lagrangian

$$g_{NN\rho}\bar{\psi}_N(x)\gamma_\mu\vec{\tau}\vec{\rho}_\mu\psi_N(x),$$

$g_{NN\rho}$ being coupling constant of nucleons to ρ mesons. Exactly as in the previous case we cannot introduce a mass for this field as in a obvious way mass term in the Lagrangian $m_\gamma \vec{\rho}_\mu \vec{\rho}^\mu$ is not invariant under the chosen gauge transformation of the field $\vec{\rho}_\mu$. Finally let us write Lagrangian invariant under the local gauge transformations of the non-abelian group $SU(2)$:

$$L = \bar{\psi}_N(x)\partial_\mu \gamma_\mu \psi_N(x) + m_N \bar{\psi}_N(x)\psi_N(x) +$$

$$g_{NN\rho}\bar{\psi}_N(x)\gamma_\mu \vec{\tau}\vec{\rho}_\mu \psi_N(x) - \frac{1}{4}\vec{F}^{\mu\nu}\vec{F}_{\mu\nu},$$

$\vec{F}_{\mu\nu}$ describing a free massless isovector field ρ. It is invariant under gauge transformations $U^\dagger \vec{F}'_{\mu\nu} U = \vec{F}_{\mu\nu}$. Let us write a tensor of free ρ meson field $\vec{\tau}\vec{F}_{\mu\nu} = \tau_k F^k_{\mu\nu} \equiv \tilde{F}_{\mu\nu}$, $\quad ([\tau_i, \tau_j] = 2i\epsilon^{ijk}\tau_k, \quad i, j, k = 1, 2, 3)$:

$$\vec{F}^k_{\mu\nu} = (\partial_\nu \rho^k_\mu - \partial_\mu \rho^k_\nu) - 2g_{NN\rho} i \epsilon^{kij} \rho^i_\mu \rho^j_\nu$$

or

$$\tilde{F}_{\mu\nu} = (\partial_\nu \tilde{\rho}_\mu - \partial_\mu \tilde{\rho}_\nu) - g_{NN\rho}[\tilde{\rho}_\mu, \tilde{\rho}_\nu]$$

and see that this expression transforms in a covariant way while gauge transformation is performed for the field ρ:

$$U^\dagger(\partial_\nu \tilde{\rho}'_\mu - \partial_\mu \tilde{\rho}'_\nu)U =$$

$$(\partial_\nu \tilde{\rho}_\mu - \partial_\mu \tilde{\rho}_\nu) + [U^\dagger \partial_\nu U, \tilde{\rho}_\mu] - [U^\dagger \partial_\mu U, \tilde{\rho}_\nu],$$

$$U^\dagger[\tilde{\rho}'_\mu, \tilde{\rho}'_\nu]U = [\tilde{\rho}_\mu, \tilde{\rho}_\nu] + \frac{1}{g_{NN\rho}}[U^\dagger \partial_\nu U, \tilde{\rho}_\mu] - \frac{1}{g_{NN\rho}}[U^\dagger \partial_\mu U, \tilde{\rho}_\nu].$$

Finally,

$$U^\dagger \vec{F}'_{\mu\nu} U = U^\dagger(\partial_\nu \tilde{\rho}'_\mu - \partial_\mu \tilde{\rho}'_\nu - g_{NN\rho}[\tilde{\rho}'_\mu, \tilde{\rho}'_\nu])U =$$

$$\partial_\nu \tilde{\rho}_\mu - \partial_\mu \tilde{\rho}_\nu + g_{NN\rho}[\tilde{\rho}_\mu, \tilde{\rho}_\nu] = \vec{F}_{\mu\nu}.$$

It is important to know the particular characteristic of the non-abelian vector field - it proves to be autointeracting, that is, in the term $(-1/4)|\vec{F}^{\mu\nu}|^2$ of the lagrangian new terms appear which are not bilinear in field ρ (as is the case for the abelian electromagnetic field), but trilinear and even quadrilinear in the field ρ, namely, $\rho_\nu \rho_\mu \partial_\nu \rho_\mu$ and $\rho^2_\nu \rho^2_\mu$.

Later this circumstance would prove to be decisive for construction of the non-abelian theory of strong interactions, that is of the quantum chromodynamics (QCD)

The Yang-Mills formalism was generalized to $SU(3)_f$ where the requirement of the local gauge-invariance of the Lagrangian describing octet baryons led to appearance of eight massless vector bosons with the quantum numbers of vector meson octet 1^- known to us.

Unfortunately along this way it proved to be impossible to construct theory of strong interactions with the vector meson as quanta of the strong field. But it was developed a formalism which make it possible to solve this problem not in the space of three flavors with the gauge group $SU(3)_f$ but instead in the space of colors with the gauge group $SU(3)_C$ where quanta of the strong field are just massless vector bosons having new quantum number 'color' named gluons.

3.4 Vector and axial-vector currents in unitary symmetry and quark model

3.4.1 General remarks on weak interaction

Now we consider application of the unitary symmetry model and quark model to the description of weak processes between elementary particles.

Several words on weak interaction. As is well known muons, neutrons and Λ hyperons decay due to weak interaction. We have mentioned muon as leptons (at least nowadays) are pointlike or structureless particle due to all know experiments and coupling constants of them with quanta of different fields act, say, in pure form not obscured by the particle structure as it is in the case of hadrons. The decay of muon to electron and two neutrinos $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ is characterized by Fermi constant $G_F \sim 10^{-5} m_p^{-2}$. Neutron decay into proton , electron and antineutrino called usually neutron $\beta-$ decay is characterized practically by the same coupling constant. However $\beta-$ decays of the Λ hyperon either $\Lambda \to p + e^- + \bar{\nu}_e$ or $\Lambda \to p + \mu^- + \bar{\nu}_\mu$ are characterized by noticibly smaller coupling constant. The same proved to be true for decays of nonstrange pion and strange K meson. Does it mean that weak interaction is not universal in difference from electromagnetic one? It may be so. But maybe it is possible to save universality? It proved to be possible and it was done more than 40 years ago by Nicola Cabibbo by introduction of the angle which naturally bears his name, Cabibbo angle θ_C.

For weak decays of hadrons it is sufficient to assume that weak interactions without change of strangeness are defined not by Fermi constant G_F but instead by $G_F cos\theta_C$ while those with the change of strangeness are defined by the coupling constant $G_F sin\theta_C$. This hypothesis has been brilliantly confirmed during analysis of many weak decays of mesons and barons either conserving or violating strangeness. The value of Cabibbo angle is $\sim 13^o$.

But what is a possible formalism to describe weak interaction? Fermi has answered this question half century ago.

We already know that electromagnetic interaction can be given by interaction Lagrangian of the type current$\times$ field:

$$L = eJ_\mu(x)A^\mu(x) = e\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x).$$

Note that, say, electron or muon scattering on electron is the process of the

2nd order in e. Effectively it is possible to write it the form current$\times$current:

$$L^{(2)} = \frac{e^2}{q^2} J_\mu J^\mu,$$

where q^2 is the square of the momentum transfer. It has turned out that weak decays also are described by the effective Lagrangian of the form current$\times$current but this has been been taken as the 1st order expansion term in Fermi constant:

$$L_W = \frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + \text{Hermitian} \quad \text{Conjugation},$$

The weak current should have the form

$$J_\mu(x) = \bar{\psi}_{\nu_\mu}(x) O_\mu \psi_\mu(x) + \bar{\psi}_{\nu_e}(x) O_\mu \psi_e(x) +$$

$$\bar{\psi}_p(x) O_\mu \psi_n(x) cos\theta_C + \bar{\psi}_p(x) O_\mu \psi_\Lambda(x) sin\theta_C.$$

The isotopic quantum numbers of the current describing neutron β decay is similar to that of the π^- meson while the current describing β decay of Λ is similar to K^- meson. Note that weak currents are charged! Since 1956 it is known that weak interaction does not conserve parity. This is one of the fundamental properties of weak interaction. The structure of the operator O_μ for the charged weak currents has been established from analysis of the many decay angular distributions and turns out to be a linear combination of the vector and axial-vector $O_\mu = \gamma_\mu(1+\gamma_5)$ what named often as $(V-A)$ version of Fermi theory of weak interaction.

Note that axial-vector couplings, those at $\gamma_\mu\gamma_5$ generally speaking are renormalized (attain some factor not equal to 1 which is hardly calculable even nowadays though there is a plenty of theories) while there is no need to renormalize vector currents due to the conservation of vector current, $\partial_\mu V_\mu = 0$.

But dimensional Fermi constant as could be seen comparing it with the electromagnetic process in the 2nd order could be a reflection of existence of very heavy W boson (vector intermediate boson in old terminology) emitted by leptons and hadrons like photon. In this case the observed processes of decays of muon, neutron, hyperons should be processes of the 2nd order in the dimensionless weak coupling constant g_W while $G_F \sim g_W^2/(M_W^2 - q^2)$, and one can safely neglect q^2.

Thus elementary act of interaction with the weak field might be written not in terms of the product *current* $\times$ *current* but instead using as a model electromagnetic interaction:

$$L = \frac{g_W}{\sqrt{2}}(J_\mu W_\mu^+ + J_\mu^\dagger W_\mu^-).$$

3.4.2 Weak currents in unitary symmetry model

And what are properties of weak interaction in unitary symmetry model? As weak currents are charged they could be related to components J_1^2 and J_1^3 of the current octet J_β^α. Comparing octet of the weak currents with the octet of mesons one can see that the chosen components of the current corresponds exactly (in unitary structure, not in space-time one) to π^- and K^- mesons. Vector current is conserved as does electromagnetic current and therefore has the same space-time structure. Then

$$V_{2\mu}^1 cos\theta_C + V_{3\mu}^1 sin\theta_C = (\bar{B}_2^\alpha \gamma_\mu B_\alpha^1 - \bar{B}_\alpha^1 \gamma_\mu B_2^\alpha) cos\theta_C$$

$$+(\bar{B}_3^\alpha \gamma_\mu B_\alpha^1 - \bar{B}_\alpha^1 \gamma_\mu B_3^\alpha) sin\theta_C$$

Here $p = B_3^1$ etc are members of the baryon octet matrix $J^P = \frac{1}{2}^+$. But for the part of currents violating parity similarly to the case of magnetic moments of baryons there are possible two tensor structures and unitary axial-vector current yields

$$-A_{2\mu}^1 = F(\bar{B}_2^\alpha \gamma_\mu \gamma_5 B_\alpha^1 - \bar{B}_\alpha^1 \gamma_\mu \gamma_5 B_2^\alpha) +$$

$$D(\bar{B}_2^\alpha \gamma_\mu \gamma_5 B_\alpha^1 + \bar{B}_\alpha^1 \gamma_\mu \gamma_5 B_2^\alpha)$$

Similar form is true for $A_{3\mu}^1$.

$$A_\mu^+ = A_{2\mu}^1 cos\theta_C + A_{3\mu}^1 sin\theta_C$$

Finally for the neutron β decay one has

$$G_A|_{n \to p+e+\bar{\nu}_e} = (F + D).$$

$$G_{pn}^A = (f + d)cos\theta_C, \quad G_{\Xi^0 \Xi^-}^A = (-F + D)cos\theta_C,$$

$$G_{p\Lambda}^A = \frac{1}{\sqrt{6}}(3F + D)sin\theta_C, \quad G_{\Lambda\Xi^-}^A = \frac{1}{\sqrt{6}}(3F - D)sin\theta_C,$$

$$G^A_{n\Sigma^-} = (-F + D)sin\theta_C, \quad G^A_{\Sigma^-\Xi^0} = (F + D)sin\theta_C,$$
$$G^A_{\Lambda\Sigma^-} = \sqrt{\frac{2}{3}} D cos\theta_C.$$

At $F = 2/3$, $\quad D = 1$ $(SU(6) \supset SU(3)_f \times SU(2)_J)$ $G_A|_{n\to p+e+\bar{\nu}_e} = \frac{5}{3}$. Experimental analysis of all the known leptonic decays of hyperons leads to $F = 0.477 \pm 0,011$, $D = 0.755 \pm 0.011$, which reproduces the experimental result $|G_A/G_V|_{n\to p+e+\bar{\nu}_e}| = 1.261 \pm 0.004$.

3.4.3 Weak currents in a quark model

Let us now construct quark charged weak currents. When neutron decays into proton (and a pair of leptons) in the quark language it means that one of the d quarks of neutron transforms into u quark of proton while remaining two quarks could be seen as spectators. The corresponding weak current yields

$$j^d_\mu = \bar{u}\gamma_\mu(1 + \gamma_5) d cos\theta_C.$$

For the Λ hyperon this discussion is also valid only here it is s quark of the Λ transforms into u quark of proton while remaining two quarks could be seen as spectators. The corresponding weak current yields

$$j^s_\mu = \bar{u}\gamma_\mu(1 + \gamma_5) s sin\theta_C.$$

Logically it comes the form

$$j_\mu = j^d_\mu + j^s_\mu = \bar{u}\gamma_\mu(1 + \gamma_5) d_C,$$

where $d_C = dcos\theta_C + ssin\theta_C$.

Thus in the quark sector (as it says now) left-handed- helicity doublet has arrived: $\begin{pmatrix} u \\ d_C \end{pmatrix}_L = (1 + \gamma_5)\begin{pmatrix} u \\ d_C \end{pmatrix}$. In the leptonic sector of weak interaction it is possible to put into correspondence with this doublet following left-handed-helicity doublets: $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$ and $\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$. (Thus the whole group theory science could be reduced to the group $SU(2)$?) Then it comes naturally an idea about existence of weak isotopic triplet of W bosons which interacts in a weak way with this weak isodoublet:

$$L = g_W \vec{j}_\mu \vec{W}_\mu + H.C.$$

(Let us remain for a moment open a problem of renormalizibility of such theory with massive vector bosons.)

Before we finish with the charged currents let us calculate constant G_A or more exact the ratio G_A/G_V for the neutron β decay in quark model. Nonrelativistic limit for the operator $\gamma_\mu\gamma_5\tau^+$ is $\sigma_z\tau^+$ where τ_+ transforms one of the d quarks of the neutron into u quark.

$$G_A^{np} =< p_\uparrow|\tau_q^+\sigma_z^q|n_\uparrow >=$$

$$= \frac{1}{6} < 2u_1u_1d_2 - u_1d_1u_2 - d_1u_1u_2|\tau^+\sigma_z^q|2d_1d_1u_2 - d_1u_1d_2 - u_1d_1d_2 >=$$

$$\frac{1}{6} < 2u_1u_1d_2 - u_1d_1u_2 - d_1u_1u_2|2u_1^*d_1u_2 + 2d_1u_1^*u_2$$

$$-u_1^*u_1d_2 + d_1u_1u_2^* - u_1u_1^*d_2 > +u_1d_1u_2^* >=$$

$$\frac{1}{6}(-2-2-2-1-2-1) = -\frac{5}{3} \quad (exp. - 1.261 \pm 0.004)$$

Quark model result coincides with that of the exact $SU(6)$ but disaccord with the experimental data that is why in calculations as a rule unitary model of $SU(3)_f$ is used.

Chapter 4

Introduction to Salam-Weinberg-Glashow model

4.1 Neutral weak currents

We are now sufficiently safe with the charged currents (and old version is quite good) but hypothesis about weak isotriplet of W leads to neutral currents in quark sector:

$$j_\mu^{neutr.,ud} = \frac{1}{2}(\bar{u}O_\mu u - \bar{d}_C O_\mu d_C) =$$

$$= \frac{1}{2}(\bar{u}O_\mu u - \bar{d}O_\mu d(cos\theta_C)^2 - \bar{s}O_\mu s(sin\theta_C)^2 -$$

$$\bar{d}O_\mu s cos\theta_C sin\theta_C - \bar{s}O_\mu d cos\theta_C sin\theta_C).$$

and, correspondingly, in lepton sector:

$$j_\mu^{neutr.,lept.} = \frac{1}{2}(\bar{\nu}_e O_\mu \nu_e - \bar{e}O_\mu e) +$$

$$+\frac{1}{2}(\bar{\nu}_\mu O_\mu \nu_\mu - \bar{\mu}O_\mu \mu).$$

(We do not write here explicitly weak neutral operator O_μ as it can diverge finally from the usual charged one $\gamma_\mu(1+\gamma_5)$.) Up to the moment when neutral currents were discouvered experimentally presence of these currents in theory was neither very intriguing nor very disturbing.

But when in 1973 one of the most important events in physics of weak interaction of the 2nd half of the XX century happened – neutral currents were discouvered in the interactions of neutrino beams of the CERN machine with the matter, it was become immediately clear the contradiction to solve: although neutral currents interacted with neutral weak boson (to be established yet in those years) with more or less the same coupling as charged currents did with the charged W bosons, there were no neutral strange weak currents which were not much suppressed by Cabibbo angle. Even more: neutral currents written above opened channel of decay of neutral K mesons into $\mu^-\mu^+$ pair with approximately the same coupling as that of the main decay mode of the charged K^- meson (into lepton pair $\mu^-\bar{\nu}_\mu$). Experimentally it is suppressed by 7 orders of magnitude!!!

$$\Gamma(K_s^0 \to \mu^-\mu^+)/\Gamma(K_s^0 \to all) < 3.2 \times 10^{-7}$$

Once more have we obtained serious troubles with the model of weak interaction!?

In what way, clear and understandable, it is possible to save it? It turns out to be sufficient to remind of the J/ψ particle and its interpretation as a state with the 'hidden' charm ($\bar{c}c$). **New quark with charm would save situation!**

4.1.1 GIM model

Indeed now the number of quarks is 4 but in the weak isodoublet only 3 of them are in action. And if one (Glashow, Iliopoulos, Maiani) assumes that the 4th quark also forms a weak isodoublet, only with the combination of d and s quarks orthogonal to $d_C = dcos\theta_C + ssin\theta_C$, namely, $s_C = scos\theta_C - dsin\theta_C$? Then apart from charged currents

$$j_\mu = \bar{c}\gamma_\mu(1+\gamma_5)s_C$$

neutral currents should exist of the form:

$$j_\mu^{neutr.,cs} = \frac{1}{2}(\bar{c}O_\mu c - \bar{s}_C O_\mu s_C) =$$

$$\frac{1}{2}(\bar{u}O_\mu u - \bar{s}O_\mu s(cos\theta_C)^2 - \bar{d}O_\mu d(sin\theta_C)^2 +$$

$$+\bar{d}O_\mu s cos\theta_C sin\theta_C + \bar{s}O_\mu d cos\theta_C sin\theta_C).$$

The total neutral current yields:

$$j_\mu^{neutr.,ud} = \frac{1}{2}(\bar{u}O_\mu u +$$

$$+\bar{c}O_\mu c - \bar{d}O_\mu d - \bar{s}O_\mu s).$$

There are no strangeness-changing neutral currents at all!

This is so called GIM mechanism proposed in 1970 by Glashow, Iliopoulos, Maiani in order to suppress theoretically decays of neutral kaons already suppressed experimentally. (For this mechanism Nobel price was given!)

4.1.2 Construction of the Salam-Weinberg model

Now we should understand what is the form of the operator O_μ. But this problem is already connected with the problem of a unification of weak and electromagnetic interactions into the electroweak interaction. Indeed the form of the currents in both interactions are remarkably similar to each other. Maybe it would be possible to attach to the neutral weak current the electromagnetic one? It turns out to be possible, and this is the main achievement of the Salam-Weinberg model.

But we cannot add electromagnetic current promptly as it does not contain weak isospin. Instead we are free to introduce one more weak- interacting neutral boson Y_μ ascribing to it properties of weak isosinglet. We shall consider only sector of u and d quarks and put for a moment even $\theta_C = 0$ to simplify discussion.

$$L = g\frac{1}{2}(\bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L)W_{3\mu} +$$

$$+g'(a\bar{u}_L\gamma_\mu u_L + b\bar{u}_R\gamma_\mu u_R + c\bar{d}_L\gamma_\mu d_L + q\bar{d}_R\gamma_\mu d_R)Y_\mu =$$

$$e[\frac{2}{3}(\bar{u}_L\gamma_\mu u_L + \bar{u}_R\gamma_\mu u_R) - \frac{1}{3}(\bar{d}_L\gamma_\mu d_L + \bar{d}_R\gamma_\mu d_R)]A_\mu +$$

$$+\kappa J_\mu^{neutr.,ud} Z_\mu^0.$$

Having two vector boson fields $W_{3\mu}, Y_\mu$ we should transfer to two other boson fields A_μ, Z^0_μ (one of them, namely, A_μ we reserve for electromagnetic field) and take into account that in fact we do know the right form of the electromagnetic current. It would be reasonable to choose orthogonal transformation from one pair of fields to another. Let it be

$$W_{3\mu} = \frac{gZ^0_\mu + g'A_\mu}{\sqrt{g^2+g'^2}}, Y_\mu = \frac{-g'Z^0_\mu + gA_\mu}{\sqrt{g^2+g'^2}}.$$

Substituting these relations into the formula for currents we obtain in the left-hand side (LHS) of the expression for the electromagnetic current the following formula

$$\frac{gg'}{\sqrt{g^2+g'^2}}[(\frac{1}{2}+a)\bar{u}_L\gamma_\mu u_L + b\bar{u}_R\gamma_\mu u_R +$$

$$+(-\frac{1}{2}c\bar{d}_L\gamma_\mu d_L + q\bar{d}_R\gamma_\mu d_R)A_\mu] = eJ^{em}A_\mu,$$

wherefrom

$$a = \frac{1}{6}, \quad b = \frac{2}{3}, \quad c = \frac{1}{6}$$

$$q = -\frac{1}{3}, \quad e = \frac{gg'}{\sqrt{g^2+g'^2}}.$$

Then for the neutral current we obtain

$$\frac{(g^2+g'^2)}{\sqrt{g^2+g'^2}}\frac{1}{2}(\bar{u}_L\gamma_\mu u_L -$$

$$-\bar{d}_L\gamma_\mu d_L)W_{3\mu} - \frac{g'^2}{\sqrt{g^2+g'^2}}J^{em} =$$

$$\frac{\sqrt{g^2+g'^2}}{g}g\frac{1}{2}(\bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L)W_{3\mu} -$$

$$-\frac{\sqrt{g^2+g'^2}}{g}g\frac{g'^2}{g^2+g'^2}J^{em}$$

Let us now introduce notations

$$sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \quad cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}.$$

Now neutral vector fields are related by formula

$$W_{3\mu} = cos\theta_W Z^0_\mu + sin\theta_W A_\mu, \quad Y_\mu = -sin\theta_W Z^0_\mu + cos\theta_W A_\mu.$$

Finally weak neutral current in the sector of u and d quarks reads

$$\frac{g}{cos\theta_W}[\frac{1}{2}(\bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L) - sin^2\theta_W J^{em}].$$

Now we repeat these reasonings for the sector of c and s quarks and restore Cabibbo angle arriving at the neutral weak currents in the model with 4 flavors:

$$J^{neutr.,GWS}_W = \frac{g}{cos\theta_W}[\frac{1}{2}(\bar{c}_L\gamma_\mu c_L + \bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L - \bar{s}_L\gamma_\mu s_L) -$$

$$-sin^2\theta_W J^{em}]. \tag{4.1}$$

Remember now that the charged current enters Lagrangian as

$$L = \frac{g_W}{2\sqrt{2}}(\bar{c}\gamma_\mu(1+\gamma_5)s_C W^+_\mu + \bar{u}\gamma_\mu(1+\gamma_5)d_C W^+_\mu +$$

$$+\bar{s}_C\gamma_\mu(1+\gamma_5)cW^-_\mu + \bar{d}_C\gamma_\mu(1+\gamma_5)uW^-_\mu)$$

and in the 2nd order of perturbation theory in ud-sector one would have

$$L^{(2)} = \frac{1}{8}\frac{g^2_W}{(M^2_W + q^2)}\bar{u}\gamma_\mu(1+\gamma_5)d_C\bar{d}_C\gamma_\mu(1+\gamma_5)u + H.C.,$$

what should be compared to

$$L^{eff} = \frac{G_F}{\sqrt{2}}\bar{u}\gamma_\mu(1+\gamma_5)d_C\bar{d}_C\gamma_\mu(1+\gamma_5)u + H.C.$$

Upon neglecting square of momentum transfer q^2 in comparing to the W-boson mass one has

$$\frac{G_F}{\sqrt{2}} = \frac{g^2_W}{8M^2_W} =$$

$$= \frac{e^2}{8M^2_W sin^2\theta_W},$$

wherefrom

$$M_W^2 \leq \frac{\sqrt{2}e^2}{8G_F} = \frac{\sqrt{2}4\pi\alpha}{8G_F} \sim 1200GeV^2$$

that is

$$M_W \geq 35GeV!!!$$

(Nothing similar happened earlier!)

Measurements of the neutral weak currents give the value of Weinberg angle as $sin^2\theta_W = 0,2311 \pm 0,0003$. But in this case the prediction becomes absolutely definite: $M_W = 73GeV$. As is known the vector intermediate boson W was discouvered at the mass $80,22 \pm 0,26$ГэВ which agree with the prediction as one must increase it by ~10% due to large radiative corrections.

4.1.3 Six quark model and CKM matrix

But nowadays we have 6 and not 4 quark flavors. So we have to assume that there is a mix not of two flavors (d and s) but of all 3 ones (d, s, b):

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

This very hypothesis has been proposed by Kobayashi and Maskawa in 1973. The problem is to mix flavors in such a way as to guarantee disappearence of neutral flavor-changing currents. Diagonal character of neutral current is achieved by choosing of the orthogonal matrix V_{CKM} of the flavor transformations for the quarks of the charge -1/3.

Even more it occurs that it is now possible to introduce a phase in order to describe violation of CP-invariance (with number of flavours less then 3 one can surely introduce an extra phase but it could be hidden into the irrelevant phase factor of one of the quark wave functions). Usually Cabibbo-Kobayashi-Maskawa matrix is chosen as

$$V_{CKM} =$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}. \quad (4.2)$$

Here $c_{ij} = cos\theta_{ij}, s_{ij} = sin\theta_{ij}, (i,j = 1,2,3)$, while θ_{ij} - generalized Cabibbo angles. At $\theta_{23} = 0, \theta_{13} = 0$ one returns to the usual Cabibbo angle $\theta_C = \theta_{12}$. Let us write matrix V_{CKM} with the help of Eqs. (1,5,7) as

$$V_{CKM} = R_1(\theta_{23}) D^*(e^{i\delta_{13}/2}) R_2(\theta_{13}) D(e^{i\delta_{13}/2}) R_3(\theta_{12}) =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta_{23} & sin\theta_{23} \\ 0 & -sin\theta_{23} & cos\theta_{23} \end{pmatrix} \begin{pmatrix} e^{-i\delta_{13}/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_{13}/2} \end{pmatrix} \times \tag{4.3}$$

$$\begin{pmatrix} cos\theta_{13} & 0 & sin\theta_{13} \\ 0 & 1 & 0 \\ -sin\theta_{13} & 0 & cos\theta_{13} \end{pmatrix} \begin{pmatrix} e^{i\delta_{13}/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta_{13}/2} \end{pmatrix} \times$$

$$\begin{pmatrix} cos\theta_{12} & sin\theta_{12} & 0 \\ -sin\theta_{12} & cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Matrix elements are obtained from experiments with more and more precision. Dynamics of experimental progress could be seen from these two matrices divided by 10 years in time:

$$V_{CKM}^{(1990)} =$$

$$= \begin{pmatrix} 0.9747 & to & 0.9759 & 0.218 & to & 0.224 & 0.001 & to & 0.007 \\ 0.218 & to & 0.224 & 0.9734 & to & 0.9752 & 0.030 & to & 0.058 \\ 0.03 & to & 0.019 & 0.029 & to & 0.058 & 0.9983 & to & 0.9996 \end{pmatrix}. \tag{4.4}$$

$$V_{CKM}^{(2000)} =$$

$$= \begin{pmatrix} 0.9742 & to & 0.9757 & 0.219 & to & 0.226 & 0.002 & to & 0.005 \\ 0.219 & to & 0.225 & 0.9734 & to & 0.9749 & 0.037 & to & 0.043 \\ 0.04 & to & 0.014 & 0.035 & to & 0.043 & 0.9990 & to & 0.9993 \end{pmatrix}. \tag{4.5}$$

The charged weak current could be written as

$$J_W^- = (\bar{u}, \quad \bar{c}, \quad \bar{t}) \gamma_\mu (1 + \gamma_5) V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Neutral current would be the following in the standard 6-quark model of Salam-Weinberg:

$$\frac{g}{\sqrt{cos\theta_W}} J_W^{нейтр.6} =$$

$$\frac{g}{\sqrt{cos\theta_W}}[\frac{1}{2}(\bar{t}_L\gamma_\mu t_L + \bar{c}_L\gamma_\mu c_L + \bar{u}_L\gamma_\mu u_L -$$

$$\bar{d}_L\gamma_\mu d_L - \bar{s}_L\gamma_\mu s_L - \bar{b}_L\gamma_\mu b_L) - sin^2\theta_W J^{em}].$$

4.2 Vector bosons W and Y as gauge fields

Bosons W and Y could be introduced as gauge fields to assure renormalization of the theory of electroweak interactions. We are acquainted with the method of construction of Lagrangians invariant under local gauge transformations on examples of electromagnetic field and isotriplet of the massless ρ-meson fields.

We have introduced also the notion of weak isospin, so now we require local gauge invariance of the Lagrangian of the left-handed and right-handed quark (and lepton) fields under transformations in the weak isotopic space with the group $SU(2)_L \times SU(1)$.

But as we consider left- and right- components of quarks (and leptons) apart we put for a moment all the quark (and lepton) masses equal to zero.

For our purpose it is sufficient to write an expression for one left-handed isodoublet and corresponding right-handed weak isosinglets u_R, d_R:

$$L_0 = \bar{q}_L(x)\partial_\mu\gamma_\mu q_L(x) + \bar{u}_R(x)\partial_\mu\gamma_\mu u_R(x) + \bar{d}_R(x)\partial_\mu\gamma_\mu d_R(x)$$

This Lagrangian is invariant under a global gauge transformation

$$q'_L(x) = e^{i\vec{\alpha}\vec{\tau}} q_L(x),$$

$$u'_{R,L}(x) = e^{i\beta_{R,L}} u_{R,L}(x),$$

$$d'_{R,L}(x) = e^{i\beta'_{R,L}} d_{R,L}(x),$$

where matrices $\vec{\tau}$ act in weak isotopic space and $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$,$\beta_{R,L}$, $\beta'_{R,L}$ are arbitrary real phases.

Let us require now invariance of this Lagrangian under similar but local gauge transformations when $\vec{\alpha}$ and $\beta_{R,L}, \beta'_{R,L}$ are functions of x. As it has been previously L_0 is not invariant under such local gauge transformations:

$$L'_0 = L_0 + i\bar{q}_L(x)\gamma_\mu \frac{\partial \vec{\tau}\vec{\alpha}(x)}{\partial x_\mu} q_L(x) +$$

$$+ i\bar{u}_R\gamma_\mu \frac{\partial \beta_R(x)}{\partial x_\mu} u_R + i\bar{d}_R\gamma_\mu \frac{\partial \beta'_R(x)}{\partial x_\mu} d_R$$

$$+ i\bar{u}_L\gamma_\mu \frac{\partial \beta_L(x)}{\partial x_\mu} u_L + i\bar{d}_L\gamma_\mu \frac{\partial \beta'_L(x)}{\partial x_\mu} d_L.$$

In order to cancel terms violating local gauge invariance let us introduce weak isotriplet of vector fields $\vec{W}_\mu$ and also weak isosinglet Y_μ with the gauge transformations

$$\vec{\tau}\vec{W}'_\mu = U^\dagger \vec{\tau}\vec{W}_\mu U - \frac{1}{g_W}\frac{\partial U}{\partial x_\mu} U^\dagger$$

где $U = e^{i\vec{\alpha}(x)\vec{\tau}}$;

$$Y'_\mu = Y_\mu - \frac{1}{g_Y}\frac{\partial(\beta_R + \beta'_R + \beta_L + \beta'_L)}{\partial x_\mu}$$

Interactions of these fields with quarks could be defined by the Lagrangian constructed above

$$L = \frac{1}{\sqrt{2}} g(\bar{u}_L\gamma_\mu d_L W^+_\mu + \bar{d}_L\gamma_\mu u_L W^-_\mu) + g\frac{1}{2}(\bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L)W_{3\mu} +$$

$$g'(a\bar{u}_L\gamma_\mu u_L + b\bar{u}_R\gamma_\mu u_R + c\bar{d}_L\gamma_\mu d_L + q\bar{d}_R\gamma_\mu d_R)Y_\mu =$$

Thus the requirement of invariance of the Lagrangian under local gauge transformations along the group $SU(2)_L \times SU(1)$ yields appearence of four massless vector fields $\vec{W}, Y$.

Earlier it has already been demonstrated in what way neutral fields $W_{3\mu}$, Y_μ by an orthogonal transformation can be transformed into the fields Z_μ, A_μ. After that one needs some mechanism (called mechanism of spontaneous breaking of gauge symmetry) in order to give masses to $W^\pm, Z$ and to leave the field A_μ massless. Usually it is achieved by so called Higgs mechanisn. Finally repeating discussion for all other flavours we come to the already obtained formulae for the charged and neutral weak currents but already in the gauge-invariant symmetry with spontaneous breaking of gauge symmetry.

4.3 About Higgs mechanism

Because of short of time we could not show Higgs mechanism in detail as an accepted way of introducing of massive vector intermediate bosons $W^{\pm}, Z^0$ into the theory of Glashow-Salam-Weinberg. We give only short introduction into the subject.

Let us introduce first scalar fields ϕ with the Lagrangian

$$L_\phi = T - V = \partial_\mu \phi \partial_\mu \phi + \mu^2 \phi^2 + \lambda \phi^4. \qquad (4.6)$$

Let us look at V just as at ordinary function of a parameter ϕ and search for the minimum of the potential $V(\phi)$.

$$\frac{dV(\phi)}{d\phi} = -2\mu^2 \phi - 4\lambda \phi^3 = 0.$$

We have 3 solutions:

$$\phi^1_{min} = 0,$$
$$\phi^{2,3}_{min} = \pm\sqrt{-\frac{\mu^2}{\lambda}}.$$

That is, with $\mu^2 \langle 0$ we would have two minima (or vacuum states) not at the zero point! How to understand this fact? Let us find some telegraph mast and cut all the cords which help to maintain it in vertical state. For a while it happens nothing. But suddenly we would see that it is falling. And maybe directly to us. What should be our last thought? That this is indeed a spontaneous breaking of symmetry!

This example shows to us not only a sort of vanity of our existence but also the way to follow in searching for non-zero masses of the weak vector bosons $W^{\pm}, Z$.

By introducing some scalar field ϕ with nonzero vacuum expectation value (v.e.v.) $< \phi >= v$ it is possible to construct in a gauge-invariant way the interaction of this scalar field with the vector bosons $W^{\pm}, W^3, Y$ having at that moment zero masses. This interaction is bilinear in scalar field and bilinear in fields $W^{\pm}, W^3, Y$ that is it contains terms of the kind $\phi^2 |W^{\pm}_\mu|^2$. At this moment the whole Lagrangian is locally gauge invariant which assures its renormalization. Changing scalar field ϕ to scalar field χ with the v.e.v. equal to zero $\chi = \phi - < \phi >, \quad < \chi >= 0$ we break spontaneously local gauge invariance of the whole Lagrangian but instead we obtain terms of the kind $v^2|W_\mu^\pm|^2$ which are immediately associated with the mass terms of the vector bosons $W_\mu^\pm$. A similar discussion is valid for Z boson.

Now we shall show this "miracle" step by step.

Firs let us introduce weak isodoublet of complex scalar fields

$$L_\phi = \partial_\mu \phi^* \partial_\mu \phi + \mu^2 \phi^* \phi + \lambda(\phi^* \phi)^2. \tag{4.7}$$

where ϕ is a doublet, $\phi^T = (\phi^+ \quad \phi^0)$ with non-zero vacuum value, $\langle \phi \rangle = v \neq 0$.

It is invariant under global gauge transformations

$$\phi' = U\phi = e^{i\vec{\alpha}\vec{\tau}}\phi.$$

Now let us as usual require invariance of the Lagrangian under the local gauge transformations. The Lagrangian Eq.(4.7) however is invariant only in the part without derivatives. So let us study "kinetic" part of it. With $\vec{\alpha}(x)$ dependent on x we have

$$\partial_\mu \phi' = U\partial_\mu \phi + \partial_\mu U \phi,$$

it becomes

$$\partial_\mu \phi^{*\prime} \partial_\mu \phi' = (\phi^* \partial_\mu U \dagger + \partial_\mu \phi^* U^\dagger)(\phi \partial_\mu U + \partial_\mu \phi \cdot U) =$$

$$= \partial_\mu \phi^* (U^\dagger U) \partial_\mu \phi + \phi^* (\partial_\mu (U^\dagger \partial_\mu U)\phi +$$
$$\phi^* (\partial_\mu U^\dagger) U \partial_\mu \phi + \phi^* U^\dagger (\partial_\mu U)\phi.$$

Let us introduce as already known remedy massless isotriplet of vector mesons $\vec{W}$ with the gauge transformation already proposed

$$\vec{\tau}\vec{W}'_\mu = U\vec{\tau}\vec{W}U^\dagger - \frac{1}{g_W}(\partial_\mu U)U^\dagger,$$

but with two interaction Lagrangians transforming under local gauge transformations as:

$$\phi^{*\prime} \tau \vec{W}_\mu^{\dagger\prime} \tau \vec{W}'_\mu \phi' = \phi^* \tau \vec{W}_\mu^\dagger \tau \vec{W}_\mu \phi +$$

$$\phi^* U^\dagger U \tau \vec{W}_\mu^\dagger U^\dagger (\partial_\mu U)\phi + \phi^* U^\dagger \partial_\mu \tau \vec{W}_\mu U^\dagger U \phi +$$

$$\phi^* \partial_\mu U^\dagger (\partial_\mu U) \phi,$$

and

$$\phi^{*\prime} \tau \vec{W}'_\mu \partial_\mu \phi' = \phi^* \tau \vec{W}_\mu \partial_\mu \phi + \phi^* \tau \vec{W}_\mu U^\dagger (\partial_\mu U) \phi.$$

The sum of all the terms results in invariance of the new Lagrangian with two introduced interaction terms under the local gauge transformations.

We can do it in a shorter way by stating that

$$(\partial_\mu + ig\tau \vec{W}'_\mu)\phi' =$$

$$(U\partial_\mu + (\partial_\mu U) + igU\tau \vec{W}_\mu U^\dagger U + (\partial_\mu U)U^\dagger)\phi =$$

$$U(\partial_\mu + ig\tau \vec{W}_\mu)\phi.$$

Then it is obvious that the Lagrangian

$$(\partial_\mu - ig\tau \vec{W}^\dagger_\mu)\phi^*(\partial_\mu + ig\tau \vec{W}_\mu)\phi$$

is invariant under the local gauge transformations.

In the same way but with less difficulties we can obtain the Lagrangian invariant under the local gauge transformation of the kind

$$\phi' = e^{i\beta}\phi$$

that is under Abelian transformations of the type use for photon previously:

$$(\partial_\mu - ig'Y_\mu)\phi^*(\partial_\mu + ig'Y\mu).\phi$$

But our aim is to obtain masses of the vector bosons. It is in fact already achieved with terms of the kind $\phi^2|W^\pm_\mu|^2$ and $\phi^2 Y^2_\mu$. Now we should also assure that our efforts are not in vain. That is searching for weak boson masses we should maintain zero for that of the photon. It is sufficient to propose the Lagrangian

$$(\partial_\mu - ig\tau \vec{W}^\dagger_\mu + ig'Y_\mu)\phi^*(\partial_\mu + ig\tau \vec{W}_\mu - ig'Y_\mu)\phi,$$

where $g^2|W^\pm_\mu|^2\phi^*\phi$ terms would yield with $\phi = \chi + v$ masses of $W^\pm$ bosons $M_W = v \cdot g$, while term $|gW^3_\mu - g'Y_\mu|^2\phi^*\phi$ would yield mass of the Z^0 boson

$$M_Z = v \cdot \sqrt{g^2 + g'^2} = v \cdot g\frac{\sqrt{g^2 + g'^2}}{g} = \frac{M_W}{cos\theta_W}.$$

There is a net prediction that the ratio M_W/M_Z is equal to $cos\theta_W$. Experimentally this ratio is (omitting errors)$\sim$ 80/91 which gives the value of Weinberg angle as $sin^2\theta_W \sim 0.23$ in agreement with experiments on neutrino scattering on protons. Due to the construction there is no term $|g'W^3_\mu + gY_\mu|^2\phi^*\phi$ that is photon does not acquire the mass!

Talking of weak bosons and scalar Higgs mesons we omit one important point that is in the previous Lagrangians dealing with fermions we should put all the fermion masses equal to zero! Why?

It is because we use different left-hand-helicity and right-hand-helicity gauge transformations under which the mass terms are not invariant as

$$m_q\bar{q}q = m_q\bar{q}_Lq_R + m_q\bar{q}_Rq_L,$$

$$q_L = \frac{1}{2}(1+\gamma_5)q, \quad q_R = \frac{1}{2}(1-\gamma_5)q.$$

What is the remedy for fermion masses? Again we could use Higgs bosons. In fact, interaction Lagrangian of quarks (similar for leptons) with the same scalar field ϕ, $\phi^T = (\pi^+ \quad \phi^0)$ with non-zero vacuum value, $\langle\phi\rangle = v \neq 0$, can be written as

$$L_{dm} = \lambda_d\bar{q}_L\phi d_R + HC = \lambda_d(\bar{u}_L\phi^+ d + \bar{d}_L\phi^0 d_R) =$$

$$\lambda_d(\bar{u}_L\chi^+ d + \bar{d}_L\chi^0 d_R + m_d\bar{d}_L d_R)$$

with $m_d = \lambda_d v$.

(In a similar way lepton masses are introduced:

$$\lambda_l(\bar{\nu}^l_L\chi^+ l + \bar{l}_L\chi^0 l_R + m_l\bar{l}_L l_R)$$

with $m_l = \lambda_l v$.)

So we could obtain now within Higgs mechanism all the masses of weak bosons and of all fermions either quarks or leptons.

By this note we finish our introduction into the Salam-Weinberg model in quark sector and begin a discussion on colour.

Chapter 5

Colour and gluons

5.1 Colour and its appearence in particle physics

Hypothesis of colour has been the beginning of creation of the modern theory of strong interaction that is quantum chromodynamics. We discuss in the beginning several experimental facts which have forced physicists to accept an idea of existence of gluons - quanta of colour field.

1) **Problem of statistics for states uuu, ddd, sss with $J^P = \frac{3}{2}^+$**

As it is known fermion behaviour follows Fermi-Dirac statistics and because of that a total wave function of a system describing half-integer spin should be antisymmetric. But in quark model quarks forming resonances $\Delta^{++} = (uuu)$, $\Delta^- = (ddd)$ and the particle $\Omega^- = (sss)$ should be in symmetric S states either in spin or isospin spaces which is prohibited by Pauli principle. One can obviously renounce from Fermi-Dirac statistics for quarks, introduce some kind of 'parastatistics' and so on. (All this is very similar to some kind of 'parapsychology' but physicists are mostly very rational people.) So it is reasonable to try to maintain fundamental views and principles and save situation by just inventing new 'colour' space external to space-time and to unitary space (which include isotopic one). As one should antisymmetrize qqq and we have the simplest absolutely antisymmetric tensor of the 3rd rank ϵ_{abc} which (as we already know) transforms as singlet representation of the group $SU(3)$ the reduction $\epsilon_{abc}q^a q^b q^c$ would be a scalar of $SU(3)$ in new quantum number called 'colour' (here $a, b, c = 1, 2, 3$ are colour indices and have no relation to previous unitary indices in mass formulae, currents and so on!) Thus the fermion statistics is saved and there is no new quantum number (like strangeness or isospin) for ordinary baryons in accord with the experimental data. But quarks become coloured and number of them is tripled. Let it be as we do not observe them on experiment.

2) **Problem of the mean life of π^0 meson**

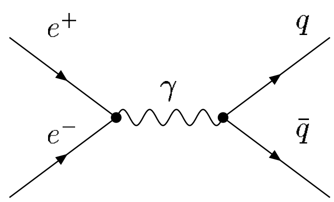
We have already mentioned that a simple model of π^0 meson decay based on Feynman diagram with nucleon loop gives very good agreement with experiment though it looks strange. Nucleon mass squared enters the denominator in the integral over the loop. Because of that taking now quark model (transfer from $m_N = 0.940$ GeV to $m_u = 0.300 GeV$ of the constituent quark) we would have an extra factor ~ 10! In other words quark model result would give strong divergence with the experimental data. How is it possible to save situation? Triplicate number of quark diagrams by introducing 'colour'!!! Really as $3^2 = 9$ the situation is saved.

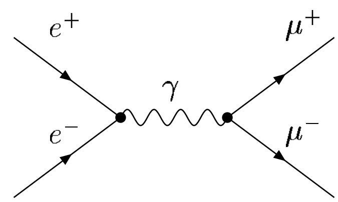
3) **Problem with the hadronic production cross-section in** e^+e^- **annihilation**

Let us consider the ratio of the hadronic production cross-section of e^+e^- annihilation to the well-known cross section of the muon production in e^+e^- annihilation:

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$

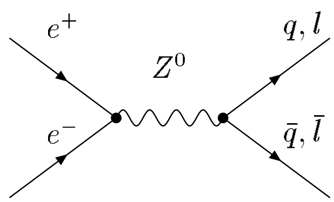
It is seen from the Feynman diagrams in the lowest order in α





that the corresponding processes upon neglecting 'hadronization' of quarks are described by similar diagrams. The difference lies in different charges of electrons(positrons) and quarks. In a simple quark model with the pointlike quarks the ratio R is given just by the sum of quark charges squared that is for the energy interval of the electron-positron rings up to 2-3 GeV it should be $R = (\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})^2) = \frac{2}{3}$. However experiment gives in this interval the value around 2.0. As one can see, many things could be hidden into the not so understandable 'hadronization' process (we observe finally not quarks but hadrons!). But the simplest way to do has been again triplication of the number of quarks, then one obtains the needed value: $3 \times \frac{2}{3} = 2$.

With the 'discouvery' of the charmed quark we should recalculate the value of R for energies higher then thresholds of pair productions of charm particles that is for energies $\geq$ 3 GeV, $R = 3 \times [(\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3}^2) + \frac{2}{3})^2)] = \frac{10}{3}$. *Production of the pair of ($\bar{b}b$) quarks should increase the value R by 1/3 which proves to hardly note experimentally. Experiment gives above 4 GeV and p to* e^+e^- *energies around 35-40 GeV the value* $\simeq 4$. *At higher energies influence on R of the Z boson contribution (see diagram) is already seen*



Thus introduction of three colours can help to escape several important and even fundamental contradictions in particle physics.

But dynamic theory appears only there arrives quant of the field (gluon) transferring colour from one quark to another and this quant in some way acts on experimental detectors. Otherwise everything could be finished at the level of more or less good classification as it succeeded with isospin and hyper charge with no corresponding quanta)

There is assumption that dynamical theories are closely related to local gauge invariance of Lagrangian describing fields and its interactions with respect to well defined gauge groups.

5.1.1 Gluon as a gauge field

Similar to cases considered above with photon and ρ meson let us write a Lagrangian for free fields of 3-couloured quarks q^a where quark $q^a, a = 1, 2, 3$

is a 3-spinor of the group $SU(3)_C$ in colour space;

$$L_0 = \bar{q}_a(x)\partial_\mu\gamma_\mu q^a(x) - m_q\bar{q}_a(x)q^a(x).$$

This Lagrangian is invariant under global gauge transformation

$$q'^a(x) = e^{i(\alpha^k\lambda^k)^a_b}q^b(x)$$

, where $\lambda^k k = 1, ...8$, are known to us Gell-Mann matrices but now in colour space. Let us require invariance of the Lagrangian under similar but local gauge transformation when α^k are functions of x:

$$q'^a(x) = e^{i(\alpha^k(x)\lambda^k)^a_b}q^b(x),$$

or

$$q'(x) = U(x)q(x), \quad U(x) = e^{i\alpha^k(x)\lambda^k}.$$

But exactly as in previous case L_0 is not invariant under this local gauge transformation

$$L_0' = \bar{q}_a(x)\partial_\mu\gamma_\mu q^a(x) + \bar{q}_a(x)(U(x)\gamma_\mu\partial_\mu U(x))^a_b(x)q^b(x)+$$

$$+m_q\bar{q}_a(x)q^a(x).$$

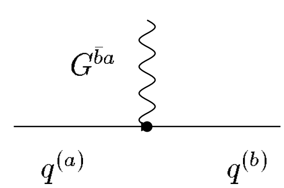
In order to cancel terms breaking gauge invariance let us introduce massless vector fields $G^k_\mu, k = 1, ...8$, with the gauge transformation

$$\lambda^k G'^k = U\lambda^k G^k U^\dagger - \frac{1}{g_s}\frac{\partial U}{\partial x_\mu}U^\dagger.$$

Let us define interaction of these fields with quarks by Lagrangian

$$L = g_s\bar{q}_a(x)(G^k_\mu\lambda_k)^a_b\gamma_\mu q^b(x),$$

to which corresponds Feynman graphs



$$G = \frac{1}{\sqrt{2}} \begin{pmatrix} G_3 + 1/\sqrt{3}G_8 & G_1 + iG_2 & G_4 + iG_5 \\ G_1 - iG_2 & -G_3 + 1/\sqrt{3}G_8 & G_6 + iG_7 \\ G_4 - iG_5 & G_6 - iG_7 & -2/\sqrt{3}G_8 \end{pmatrix} =$$

$$= \begin{pmatrix} \mathcal{D}_1 & G^{\bar{2}1} & G^{\bar{3}1} \\ G^{\bar{1}2} & \mathcal{D}_2 & G^{\bar{2}3} \\ G^{\bar{1}3} & G^{\bar{2}3} & \mathcal{D}_3 \end{pmatrix}, \qquad (5.1)$$

where

$$\mathcal{D}_1 = \frac{1}{2}(G^{\bar{1}1} - G^{\bar{2}2}) + \frac{1}{6}(G^{\bar{1}1} + G^{\bar{2}2} - 2G^{\bar{3}3});$$

$$\mathcal{D}_2 = -\frac{1}{2}(G^{\bar{1}1} - G^{\bar{2}2}) + \frac{1}{6}(G^{\bar{1}1} + G^{\bar{2}2} - 2G^{\bar{3}3});$$

$$-\frac{2}{6}(G^{\bar{1}1} + G^{\bar{2}2} - 2G^{\bar{3}3}).$$

Final expression for the Lagrangian invariant under local gauge transformations of the non-abelian group $SU(3)_C$ in colour space is:

$$L_{SU(3)_C} = \bar{q}_a(x)\partial_\mu\gamma_\mu q^a(x) + m_q\bar{q}_a(x)q^a(x) +$$

$$g_s\bar{q}_a(x)(G^k_\mu\lambda_k)^a_b\gamma_\mu q^b(x) - \frac{1}{4}\vec{F}^{k\mu\nu}F^k_{\mu\nu},$$

where $F^{k\mu\nu}, k = 1, 2, ...8$, is the tensor of the free gluon field transforming under local gauge transformation as

$$\lambda_k F^{k\prime}_{\mu\nu} = U^\dagger(x)\lambda_l(x)F^l_{\mu\nu}U(x).$$

It is covariant under gauge transformations $U^\dagger\vec{F}'_{\mu\nu}U = \vec{F}_{\mu\nu}$. Let us write in some detail tensor of the free gluon field $\vec{\lambda}\vec{F}_{\mu\nu} = \lambda_k F^k_{\mu\nu} \equiv \tilde{F}_{\mu\nu}, \quad ([\lambda_i, \lambda_j] = 2i\epsilon^{ijk}\lambda_k, \quad i, j, k = 1, 2, ...8)$:

$$\vec{F}^k_{\mu\nu} = (\partial_\nu G^k_\mu - \partial_\mu G^k_\nu) - 2g_s i f^{kij} G^i_\mu G^j_\nu$$

or

$$\tilde{F}_{\mu\nu} = (\partial_\nu\tilde{G}_\mu - \partial_\mu\tilde{G}_\nu) - g_s[\tilde{G}_\mu, \tilde{G}_\nu]$$

and prove that this expression in a covariant way transforms under gauge transformation of the field G:

$$U^\dagger(\partial_\nu \tilde{G}'_\mu - \partial_\mu \tilde{G}'_\nu)U =$$

$$= (\partial_\nu \tilde{G}_\mu - \partial_\mu \tilde{G}_\nu) + [U^\dagger \partial_\nu U, \tilde{G}_\mu] - [U^\dagger \partial_\mu U, \tilde{G}_\nu],$$

$$U^\dagger[\tilde{G}'_\mu, \tilde{G}'_\nu]U = [\tilde{G}_\mu, \tilde{G}_\nu] + \frac{1}{g_s}[U^\dagger \partial_\nu U, \tilde{G}_\mu] - \frac{1}{g_s}[U^\dagger \partial_\mu U, \tilde{G}_\nu].$$

Finally

$$U^\dagger \vec{F}'_{\mu\nu} U = U^\dagger(\partial_\nu \tilde{G}'_\mu - \partial_\mu \tilde{G}'_\nu - g_s[\tilde{G}'_\mu, \tilde{G}'_\nu])U =$$

$$= \partial_\nu \tilde{G}_\mu - \partial_\mu \tilde{G}_\nu + g_s[\tilde{G}_\mu, \tilde{G}_\nu] = \vec{F}_{\mu\nu}.$$

The particular property of non-Abelian vector field as we have already seen on the example of the ρ field is the fact that this field is autointeracting that is in the Lagrangian in the free term $(-1/4)|\vec{F}^{\mu\nu}|^2$ there are not only terms bilinear in the field G as it is in the case of the (Abelian) electromagnetic field but also 3-and 4- linear terms in gluon field G of the form $G_\nu G_\mu \partial_\nu G_\mu$ and $G_\nu^2 G_\mu^2$ to which the following Feynman diagrams correspond:

This circumstance turns to be decisive for construction of the non-Abelian theory of strong interaction - quantum chromodynamics.

The base of it is the asymptotic freedom which can be understood from the behaviour of the effective strong coupling constant of quarks and gluons $\alpha_s = g_s^2/4\pi$ for which

$$\alpha_s(Q^2) \sim \frac{\alpha_s \mu^2}{1 + (11N_C - 2n_f)\ln(Q^2/\Lambda^2)},$$

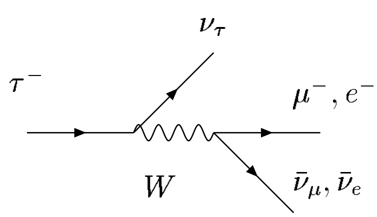
where Q^2 is momentum transfer squared, μ^2 is a renormalization point, Λ is a QCD scale parameter, N_C being number of colors and n_f number of flavours. With Q^2 going to infinity coupling constant α_s tends to zero! Just this property is called asymptotic freedom.(Instead in QED (quantum electrodynamics) with no colors it grows and even have a pole.) But one should also have in mind that in the difference from QED where we have two observable quantities electron mass and its charge (or those of μ- and τ-leptons) in QCD we have none. Indeed we could not measure directly either quark mass or its coupling to gluon.

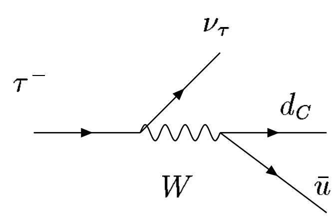
Here we shall not discuss problems of the QCD and shall give only some examples of application of the notion of colour to observable processes.

5.1.2 Simple examples with coloured quarks

By introducing colour we have obtained possibility to predict ratios of many modes of decays and to prove once more validity of the hypothesis on existence of colour.

Let us consider decay modes of lepton τ discouvered practically after J/ψ which has the mass ~1800 MeV (exp.(1777,1 +0,4-0,5) MeV). Taking quark model and assuming pointlike quarks (that is fundamental at the level of leptons) we obtain that τ^- lepton decays emitting τ neutrino ν_τ either to lepton (two channels, $e^-\bar{\nu}_e$ or $\mu^-\bar{\nu}_\mu$) or to quarks (charm quark is too heavy, strange quark contribution is suppressed by Cabibbo angle and we are left with u and d quarks).

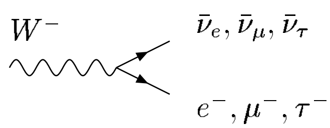


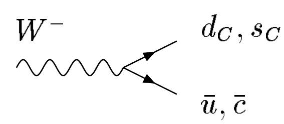


From our reasoning it follows that in absence of color we have two lepton modes and only one quark mode and partial hadron width B_h should be equal to 1/3 of the total width while with colour quarks we have two lepton modes and three quark modes leading to $B_h \sim 3/5$.

Or in other words definite lepton mode would be 33 % in absence of colour and 20 % with the colour. Experiment gives $\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau = (17,37 \pm 0,09)\%$ and $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau = (17,81 \pm 0,07)\%$, supporting hypothesis of 3 colours.

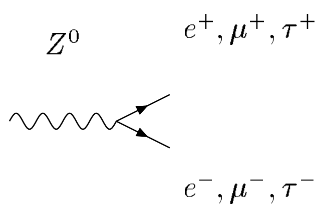
The W decays already in three lepton pairs and two quark ones

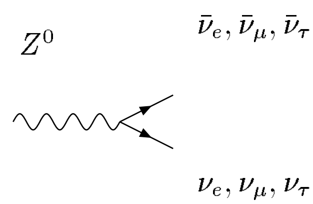


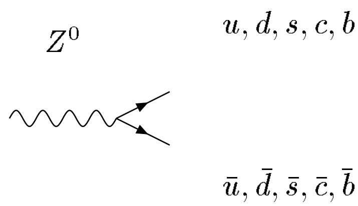


This means that in absence of colour hadron branching ratio B_h would be 40% while with three colours around 66%. Experiment gives $B_h \sim (67,8 \pm 1,5)\%$ once more supporting hypothesis of 3-coloured quarks.

Z boson decays already along 6 lepton and 5 quark modes,







Thus it is possible to predict at once that in absence of colour hadron channel should be $5/11 \simeq 45\%$ of the total width of Z while with three colours number of partial lepton and colour quark channels increase up to $6+3\times5=21$, and hadron channel would be $15/21 \simeq 71\%$. Experimentally it is $(69.89 \pm 0.07\%)$.

Chapter 6

Conclusion

In these chosen chapters on group theory and its application to the particle physics there have been considered problems of classification of the particles along irreducible representations of the unitary groups, have been studied in some detail quark model. In detail mass formulae for elementary particles have been analyzed. Examples of calculations of the magnetic moments and axial-vector weak constants have been exposed in unitary symmetry and quark model. Formulae for electromagnetic and weak currents are given for both models and problem of neutral currents is given in some detail. Electroweak current of the Glashow-Salam-Weinberg model has been constructed. The notion of colour has been introduced and simple examples with it are given. Introduction of vector bosons as gauge fields are explained.

Author has tried to write lectures in such a way as to give possibility to eventual reader to evaluate by him- or herself many properties of the elementary particles.